Equilibrium pricing, default and contagion

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Summary

We consider the impact of equilibrium requirements on (i) asset prices and (ii) default in exchange economies where agents maximize an expected-utility functional and stochastic endowments are specified exogenously.

In the first part of the thesis, we study a complete market containing several assets, each asset contributing to the production of a single commodity at a rate which is a solution to the Cox-Ingersoll-Ross (CIR) SDE. The assets are owned by agents with CRRA utility functions who follow feasible consumption/investment regimes so as to maximize their expected lifetime utility from consumption. We compute the equilibrium for this economy and determine the state-price density process from market clearing. Reducing to a single (representative) agent, and exploiting the relation between the CIR and squared-Bessel SDE's, we obtain closed-form expressions for the values of bonds and assets. We fit the model to bond price data, price assets and options on the total asset value, and estimate implied volatility surfaces.

The second part presents an equilibrium model with the potential to generate endogenous dependence between defaults of firms. Agents owning shares in firms are entitled to an aggregate stochastic cashflow that is not restricted to be positive, while limited liability allows agents to give up ownership of firms whose output is deemed to be too low. The objective of agents is to maximize expected lifetime utility of consumption; in equilibrium this may lead to complex dependence, and even contagion, among defaults of different firms. The basic model constitutes one firm and a representative agent with CARA utility, and the solution for this case can be given when the aggregate cashflow is as general as a Lévy process. The features of the two-firm model with special choices of dynamics for firm output are examined numerically.

Acknowledgements

"For seven and a half million years,
Deep Thought computed and calculated,
and in the end announced that the answer
was in fact Forty-two - and so another, even bigger,
computer had to be built to
find out what the actual question was".

It is sobering to realise that my own 'seven and a half million years' are now at an end, and while many actual questions remain I should hope that more than the number Forty-two has been achieved. I am grateful to the many people who have supported me in many ways.

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Chapter I

Introduction

The work presented in this thesis considers the implications of economic equilibrium for (i) the pricing of assets and (ii) the occurrence of default in concrete models that admit quite explicit solutions. We take agent preferences and agent endowments as our model ingredients and characterize in terms of them the equilibrium that arises when agents maximize an expected-utility functional.

Chapter II studies in detail an economy where the aggregate stochastic output is a so-called squared Ornstein-Uhlenbeck (Cox-Ingersoll-Ross (CIR)) diffusion process, this being the sum total of production from a number of independent firms. Agents owning shares in the firms are entitled to flows of the good and aim to maximize expected utility from consumption over an infinite lifetime. In equilibrium, markets clear and share prices adjust so that all output of the good is consumed. With a judicious choice for the representative agent utility in the economy - one of CRRA form - the special structure of the CIR diffusion process allows us to derive explicit expressions for prices of bonds, shares, and options on the total market asset, in this equilibrium. These prices are in terms of special functions that can, with some effort, be computed numerically. In particular, the asymptotic behaviour of the total market asset when the output of the economy is small can be exhibited. A novel form of dynamics for the endogenous riskless rate arises also.

Because of the simplicity of this model and the availability of closed-form pricing expressions *in terms of the primitives of the economy*, we were able to calibrate the model from bond price data. The fitted model parameters, in particular the risk aversion coefficient and subjective discount rate, are sensible. We also find that dynamics for share prices have features in common with the well-known family of models with constant elasticity of variance[1].

The Markov structure of our model means that the equilibrium price for the total asset and the equilibrium state-price density are deterministic functions of the aggregate consumption process, Δ. From a mathematical point of view, our explicit pricing expression for the total market asset arises precisely because the action of the resolvent of the CIR diffusion on functions of form $x \mapsto xU'(x)$, where U is the representative utility, can be computed in terms of confluent hypergeometric functions. It is to be expected that tractable expressions will be available also in other examples as long as the process Δ and the representative utility U are chosen so that the resolvent operation can be evaluated[2].

In taking the representative utility as our starting point, we are treating essentially a one-agent market (see, however, the remarks in Section II.8). In a multi-agent market without frictions, equilibrium is Pareto-efficient and allocations of the aggregate consumption is such that agents' marginal utilities differ by at most multiplicative constants. These constants will be such that individual agents' budget constraints are binding. A relation such as (II.3.3) then holds, with the equilibrium state-price density being exactly the marginal representative utility evaluated at the aggregate consumption, See, for example, Constantinides (1982), Huang (1987), and Karatzas, Lehoczky and Shreve (1990). A particular example, claimed by the last three authors to be tractable, is worked out by Wang (1995).

Although it will not concern us directly here, there has been much recent research into characterizing equilibria where there are consumption or portfolio constraints on some agents, in which case the equilibrium allocation of the aggregate consumption is no longer Pareto-efficient. Basak and Cuoco (1998), for example, study a two-agent model with one agent investing in a bond and a share and another agent restricted to keep all his wealth in the bond. They exhibit an equilibrium in which the ratio of the agents' state-price densities is stochastic rather than constant; with appropriate specific choices for the aggregate consumption they obtain useful conclusions about the behaviour of the equilibrium spot rate and asset price. In a similar framework, Basak (1995) obtains the representative

[1] See, for example, Schroder(1989), for a description of these models.

[2] In particular, log-Brownian asset prices would arise from a representative utility of CRRA form together with log-Brownian aggregate consumption. This of course concurs with the model studied by Merton (1969), who obtained his solution by exploiting scaling properties of the HJB optimality equation.

utility when the second agent *is* allowed to trade in the share but has to have wealth that exceeds a given level at a particular fixed time. Detemple and Serrat (2003) derive the form of the equilibrium state-price density in a two-agent market where one of the agents has restricted liquidity in that he is prevented from trading his labour income.

Chapter III presents an equilibrium model for default of firms, in which we view firms as an entitlement to a cashflow that is not restricted to be positive. Shareholders reap dividends when firm outputs are large and positive, and may even shore up firms by injecting capital in expectation of higher future dividends. However, we allow shareholders to decide to give up aggregate ownership of any share, thereby replacing its output by a cashflow that is identically zero; this decision is made with the intent that in equilibrium agents maximize infinite-lifetime utility of consumption.

The motivation for such a model is two-fold. First, an equilibrium analysis offers prospects of understanding *why* default events come to depend on each other, and at a more general level how assets may lend value to each other beyond what is explainable solely by correlation. The question is somewhat side-stepped in the extant credit risk literature, which has as its main goal the pricing of credit instruments in a way consistent with observed data. Second, the effect on equilibrium of a firm's default may well be to immediately cause default of another firm. In other words, equilbrium can yield an explanation of financial contagion phenomena.

In the credit-derivatives business, of course, modelling the dependence between default events is becoming increasingly crucial for pricing basket derivatives exposed to default risk. Most approaches involve extending the intensity-based paradigm to several underlying factors. Jarrow and Yu (2001), for example, divide entities into two types and let default intensities for one type to jump upwards upon default of an entity of second type. Schönbucher and Schubert (2001) use a copula function to introduce dependency between defaults modelled as jump times of Cox processes. Interestingly, Giesecke (2002, 2003) works in a structural model and proposes an information-based model that admits simultaneous defaults that arise endogenously. Our main concern in this work will be to investigate the form of dependence between defaults that arises *endogenously* when agents exercise limited liability and behave rationally.

We consider two versions of the model, which we set out in Section III.2. In discrete time, we assume that firms' output is a sequence of IID random vectors. In continuous time, we model cashflows from firms as Lévy processes. Again, we treat only the representative agent's problem in detail, assuming that the representative utility is of CARA form, but in Appendix A.4 we show how this can be made consistent with a multi-agent market. The model with one firm admits a fairly explicit solution, and details of this are worked out in Sections III.3 and III.4. Even with just two firms, however, one has to resort to numerical solutions, and we present these in Sections III.5 and III.6.

Chapter II

The Squared Ornstein-Uhlenbeck Market

Apart from slight additions and amendments, this Chapter has appeared as Aquilina and Rogers (2004).

II.1. Introduction

Characterizations of equilibrium prices in market models with intertemporal consumption abound. Unfortunately, however, the computation of these equilibria, or the application of the pricing mechanisms themselves (e.g. solutions to partial differential equations, or fixed points of certain operators) is not only not straightforward but in most cases outright impossible unless several simplifications are made. In this Chapter, we study an equilibrium model that is simple enough that equilibrium pricing expressions become explicit (and involve at most numerical integration), but also rich enough that these prices possess interesting behaviour that one can study.

We compute in detail the equilibrium for an economy having as primitives (i) shares in firms that produce a single good paid out in the form of dividends modelled by diffusion processes, and (ii) market agents who consume the good and who trade shares at market prices in order to maximize expected time-additive utility of lifetime consumption. In equilibrium markets clear, and share and bond prices adjust so that all output of the good is consumed and shares of the firms are in unit net supply.

By making judicious choices for a representative agent utility and for the dividend processes in (i) above, we derive explicit expressions for the equilibrium prices of bonds and for investing in the firms. In particular, we assume a representative utility of constant relative risk aversion (CRRA) form, and that the flows of the good from the different firms occur at rates that are independent Cox-Ingersoll-Ross (CIR) diffusion processes (see Cox, Ingersoll and Ross (1985b), hereafter referred to as CIR 1985b). Importantly, under the latter assumption, the aggregate flow is itself a CIR process. Our model is simple: we do not model the supply side of the economy[1], endowment consists of only one good, we solve only the representative agent's problem, and our market is automatically complete because firms' production is driven by independent Brownian Motions. Because of this simplicity (or in spite of it !) our model allows us to characterize the equilibrium total market value of output of the good as a diffusion with interesting properties (one whose asymptotic behaviour we can exhibit analytically, for instance). We also obtain endogenously a one-factor model for the real interest rate that is new, so far as is known to us.

In general terms, our methodology is close in spirit to that of Karatzas, Lehoczky and Shreve (1990) (hereafter referred to as KLS 1990) and Duffie and Zame (1989). In these papers, the idea is to derive the market state-price density from a representative agent's marginal utility evaluated at the aggregate consumption level, which is equal to the total output of the economy by market clearing. Prices of all market assets then follow from martingale representation with respect to gains processes of the assets. Such pricing formulas were obtained by Lucas (1978) in a discrete-time setting, but his approach was to study fixed points of the Bellman operator of a dynamical program, rather than martingale representation. Considering only one market agent, he derives explicit prices only for cases where one asset pays out a sequence of independent identically distributed dividends, or where agents are identical with linear utility. Aase (2002) derives equilibrium interest rates in models with CRRA (respectively CARA) utility agents and lognormal (respectively Gaussian) endowments with constant coefficients. As expected, the interest rates are constant. In similar settings, Aase (2002)

[1]The processes in (i) can be thought of as the output at equilibrium from firms in which optimally behaving market agents invest labour as well as units of the good. See section 2 of Breeden (1979) for more on this.

also obtains Black-Scholes-like equilibrium pricing formulas for call options on dividend-paying assets. 'Explicit' equilibrium solutions are given in KLS (1990) for a market with identical agents having power-law utility, and for a heterogeneous market where the aggregate production is a constant. These authors also claim that a two-agent equilibrium for agents with logarithmic and square-root utilities can be computed. This is true only because for such a pair of utilities, the state price density solves a quadratic equation, and even then, computing the equilibrium weights involves taking expectations of a complicated function of the aggregate endowment process.

As the examples mentioned above show, the difficulty of computing explicit equilibria in a multi-agent economy is well-appreciated. The representative agent utility for such an economy is a weighted sum of the individual agents' utilities. Proving equilibrium is tantamount to exhibiting weights that correspond to the individual agents attaining their optimal levels of consumption. This issue is what KLS (1990) deals with; its resolution depends on fixed-point arguments which are non-constructive. In practice, if any of this is to be done explicitly, the only way to solve for the representative agent weights is to render them irrelevant by studying a market with identical agents; this is what we do here, and this is what is done in all examples assuming a one-agent market. In contrast to the examples already mentioned, however, our model yields prices which are far from trivial and which exhibit interesting characteristics. Our expression for the interest spot rate, Equation (II.4.5), and our pricing recipe (Equation (II.3.2)) are consistent with the general formulas in Duffie and Zame (1989), KLS (1990), and also Aase (2002).

Duffie and Huang (1985) and Duffie (1986) were the first to apply martingale representation technology to show how the pricing function in the classic Arrow-Debreu equilibrium (see, e.g. Arrow and Debreu 1954) can be characterized as an expectation. By this important result, equilibrium is attainable by trading in a finite number of market securities. Indeed, the driving force behind the general pricing relations obtained in KLS (1990, Th. 8.2) is martingale representation with respect to gains processes of productive assets.

Building on Duffie and Huang (1985) and Duffie (1986), Huang (1987) showed how equilibrium is consistent with a representative agent who, endowed with the aggregate dividend output from a set of market securities maximizes expected time-additive utility from consumption. He also proved that if the consumption process attains its essential infimum only on a set of measure zero, then each individual agent's optimal consumption is a smooth function of the aggregate consumption. This is key to applying the fixed-point arguments employed by KLS (1990), as becomes very clear in the examples considered by these authors.

While we can prove explicitly what the equilibrium is for our market, our model primitives do not concord with several conditions that are sufficient for the general results in KLS (1990). For example, in KLS (1990), the aggregate endowment process in the economy is assumed to have bounded diffusion coefficients, an assumption which we do not make. Also, the martingale change of measure is forced to be well behaved by the restrictive condition that its diffusion coefficient be bounded. This condition does not hold for our specific model; in fact our martingale change of measure is unbounded, but a simple criterion involving the parameters of the model ensures that the martingale property still goes through (see Appendix A.1). Finally, in the KLS (1990) model, agents' utilities satisfy a condition that is equivalent to the relative risk aversion coefficients of the agents (which vary with their optimal consumption processes) being bounded above by 1. This condition is required to ensure uniqueness of equilibrium. The CRRA utility functions that we consider have (constant) risk aversion coefficient R, and we require only that R be positive. Uniqueness of equilibrium in our model is proved explicitly. In fact, in a simplified version of the KLS (1990) paper, Karatzas et al. (1991), CRRA utilities are singled out as a class of utilities for which the quite restrictive condition on the risk aversion is not necessary for uniqueness.

In a celebrated paper, Cox, Ingersoll and Ross (1985a) (hereafter cited as CIR 1985a) develop an equilibrium model for a production economy. In their model, technological change in production is modelled by a state variable, and consumption depends on the model uncertainty only through this state variable. There is a single good, and this can be consumed or invested in one of several production processes whose output depends on the technology state variable as well as amount of good invested. Equilibrium in this model involves choosing levels of investment that maximize a given expected time-additive utility for consumption. Within this framework, CIR (1985a) obtain expressions for the equilibrium rate of interest and for the optimal rate of return from production, and also derive a differential equation that prices of contingent claims must satisfy. Because their model is based on returns in raw production, rather than on the market price for shares in the production process, their budget equation differs from ours. However, in their model, the value of a share in a firm that invests in production can also be viewed as a claim to a dividend stream flowing at rate equal to the rate of return for the firm's investment (with the difference that shares are in net positive supply). The price of a firm that invests in production therefore satisfies the pricing differential equation; at equilibrium, the firm's value must equal the value of the supply of the good that the firm owns. In this sense, asset prices as derived in our model are implicit in the differential equations given by CIR (1985a). The viewpoint of Sundaresan (1984) is more aligned with ours; his budget equation involves market prices of assets rather than wealth invested in production. For the special and simple case of production returns with Cobb-Douglas drifts and constant volatility, and zero technological change, he derives simple expressions for the interest rate at equilibrium.

CIR (1985b) obtained their well-known model for term structure of interest rates by specializing the model presented in CIR (1985a). To do this, they assume that the single state variable (technological change) is a CIR (squared Ornstein-Uhlenbeck) diffusion[2] and that the means and variances of the rates of return on production are proportional to the level of the state process. There is one agent with logarithmic utility that depends on the state variable only through consumption; from this, the equilibrium spot rate process is determined to be also a CIR diffusion. By this, the density function for the law of the spot rate can be written in closed form, allowing CIR (1985b) to derive closed-form prices for bonds and for options thereon. All the analysis in our model also hinges on knowledge of the CIR diffusion, in particular its relation to the squared Bessel process.

The rest of this Chapter is organized as follows. Section II.2 describes the primitives of our economy, for which we derive an explicit equilibrium in Section II.3. In Section II.4, we derive the equilibrium spot rate and martingale change of measure for the particular case of a single representative agent with CRRA utility. As described above, the development is quite conventional, and has much in common with KLS (1990), Aase (2002) and Breeden (1979); the state-price density process is determined from market clearing, and assets are priced from that. There is nothing particularly new here at a general theoretical level, but our model assumptions are sufficiently specific that we have the rare pleasure of being able to compute various prices *in closed form.*

[2]This is the same process that the *consumption flows* in our model follow.

Section II.5 contains some computations involving squared Bessel processes, which we then use in Section II.6 to obtain expressions for the prices of bonds, assets and options in the market. In Section II.7 we calibrate the model using observed bond price data, and use typical parameters to evaluate and study numerically the market prices of assets and of options on the total asset.

II.2. The Model

We consider a market containing one unit of each of J productive assets, the j'th of which produces the single commodity of the economy at rate $\delta^j \equiv (\delta^j_t)_{t\geq 0}$ satisfying the (Cox-Ingersoll-Ross) SDE[3]

$$d\delta^j_t = \sigma\sqrt{\delta^j_t}dW^j_t + (a_j - \beta\delta^j_t)dt, \quad 1 \leq j \leq J. \tag{II.2.1}$$

Here, $\sigma > 0$, $\beta > 0$, and $a_j > 0$ are constants and $W \equiv (W^j)_{1\leq j\leq J}$ is a standard J-dimensional Brownian motion.[4] We shall write $\delta = (\delta^j)_{1\leq j\leq J}$ for the $\mathbb{R}^J$-valued rate-of-production process. *Note that the processes δ^j are independent, with common volatility parameter σ and mean-reversion parameter β.* These assumptions are restrictive but essential; the smallest variation destroys the mathematical analysis which leads to our closed-form expressions (though of course the economic structure of the solution remains unaltered.) The crucial point is the following: *the total production rate $\Delta \equiv (\Delta_t := \sum_{j=1}^J \delta^j_t)_{t\geq 0}$ satisfies an SDE of a type similar to (II.2.1):*

$$d\Delta_t = \sigma\sqrt{\Delta_t}dB_t + (A - \beta\Delta_t)dt, \tag{II.2.2}$$

where $A = \sum_{j=1}^J a_j$ and the one-dimensional Brownian Motion B is related to W via $\sqrt{\Delta_t}dB_t = \sum_{j=1}^J \sqrt{\delta^j_t}dW^j_t$. We shall maintain the standing assumption[5] :

$$\frac{2A}{\sigma^2} \geq 1. \tag{II.2.3}$$

[3]The SDE (II.2.1) falls just within the scope of the Yamada-Watanabe conditions for pathwise uniqueness of solutions. Because a weak solution exists, and pathwise uniqueness holds, the SDE is exact.

[4]The probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is the usual augmentation of the filtration generated by W - see, for example, Rogers and Williams (2000).

[5]... equivalent to the statement that $\mathbb{P}(\Delta_t > 0 \quad \forall t > 0) = 1$ ...

The assets in the market are owned by K agents; agent k has C^1 utility function $U_k : (0,\infty) \times (0,\infty) \to \mathbb{R}$, which is increasing and strictly concave in its second argument, and satisfies the Inada conditions at 0 and ∞. Agent k begins with $\alpha_k^j(0)$ units of asset j, and aims to consume according to a feasible non-negative process $c_k \equiv (c_k(t))_{t\geq 0}$ so as to maximize the objective function

$$\mathbb{E}\bigg[\int_0^\infty U_k(t, c_k(t))dt\bigg], \quad 1 \leq k \leq K. \tag{II.2.4}$$

In the next Section, we will compute the equilibrium for this economy. Our SDE (II.2.2) is closely related to the squared-Bessel SDE (see Revuz and Yor (2001) for the most important facts on these); in particular, it is possible to obtain a closed-form expression for the transition density of the diffusion δ, and this is the key to the various pricing expressions we will derive in Section II.6.

II.3. Equilibrium

Suppose that at time t agent k consumes at rate $c_k(t)$. His wealth may be invested in the assets available on the market, or in a riskless bank account bearing interest at instantaneous rate r_t. If $\alpha_k = (\alpha_k^j(t))_{t\geq 0}$, $j = 1, \dots, J$, denotes the (J-vector) process of his holdings of the asset, then his wealth $X_k = (X_k(t))_{t\geq 0}$ will evolve according to the dynamics

$$\begin{aligned} dX_k(t) = r_t X_k(t)dt + \alpha_k(t)\cdot[dS_t - r_t S_t dt] + [\alpha_k(t)\cdot\delta_t - c_k(t)]dt; \\ X_k(0) = \alpha_k(0)\cdot S_0. \end{aligned} \tag{II.3.1}$$

Here, S is the (J-vector) price-process of the assets; this and the instantaneous rate of interest r are *a priori* unknown, but will be obtained from equilibrium considerations. The only constraint on the agent's investment and consumption decisions is that his wealth X_k should remain non-negative at all times (to prevent him consuming unboundedly by running up ever larger debts.) If c_k^* denotes the optimal[6] consumption process for agent k with objective (II.2.4), wealth dynamics (II.3.1) and the non-negativity constraint on wealth, then agent k's marginal price

[6]... assumed for the moment to exist: this point will be dealt with later for our explicit example.

for a cashflow $(\varepsilon_t)_{t\geq 0}$ is simply given by

$$\mathrm{E}\left[\int_0^\infty \zeta_k(t)\varepsilon_t\, dt\right]/\zeta_k(0), \tag{II.3.2}$$

where $\zeta_k(t) \equiv U_k'(t, c_k^*(t))$ is agent k's state-price density process: see, for example, KLS (1990). Since we have a complete market, all agents will value a given cashflow the same, which implies that the state-price density processes are all multiples of one another: for some constants λ_k,

$$U_k'(t, c_k^*(t)) = \lambda_k \zeta_t \quad \forall k,\ \forall t. \tag{II.3.3}$$

Turning this around, we have for all k that

$$c_k^*(t) = I_k(t, \lambda_k \zeta_t), \quad 1 \leq k \leq K, \tag{II.3.4}$$

where I_k is the inverse marginal utility, defined by $U_k'(t, I_k(t,x)) = x$ for all $x > 0$, $t \geq 0$. On the other hand, we have the *market clearing condition*, that all of the commodity must be exactly consumed as it is produced. Thus

$$\sum_{k=1}^K c_k(t) = \sum_{j=1}^J \delta^j(t) = \Delta_t, \tag{II.3.5}$$

which together lead to the relation

$$\Delta_t = \sum_{k=1}^K I_k(t, \lambda_k \zeta_t). \tag{II.3.6}$$

If the constants λ_k were known, then, this equation (II.3.6) would determine the state-price density ζ (and hence market prices of all cashflows) from the data Δ and from the agents' preferences. For example, and in particular, the asset prices (which appeared in (II.3.1) as unknowns) would be given by

$$\zeta_t S(t) = \mathbb{E}\left[\int_t^\infty \zeta_u \delta(u) du \middle| \mathcal{F}_t\right] \tag{II.3.7}$$

In general, the constants λ_k are determined from the initial wealths of the agents, but it seems in practice that virtually the only case where we can solve for the constants is in the case of a single representative agent.

II.4. Representative agent equilibrium

In this Section, we restrict ourselves to the case ($K = 1$) where there is only one agent in the market. We can think of this as a market with all agents having identical utilities, or as a market where one representative agent acts as proxy for all the agents. We further assume[7] that there are positive constants $R \neq 1$ and ρ in terms of which $U(t,x) = e^{-\rho t}x^{1-R}/(1-R)$, so that

$$U'(t,x) = e^{-\rho t}x^{-R}. \tag{II.4.1}$$

The special case $R = 1$ corresponds of course to logarithmic utility.

Dropping the now irrelevant subscripts k that previously identified the agents, in equation (II.3.6) we have $c^*(t) = \Delta_t$, and the equation (II.3.3) then reduces to

$$e^{-\rho t}\Delta_t^{-R} = \zeta_t, \tag{II.4.2}$$

where we have without loss of generality taken $\lambda_1 = 1$. Thus we have ζ_t explicitly as a smooth function of t and Δ_t; applying Itô's lemma to ζ from equation (II.4.2) therefore gives

$$d\zeta_t = \zeta_t\Big[\frac{-R\sigma}{\sqrt{\Delta}}dB_t - \{\rho - \beta R + \frac{R(A-\sigma^2(R+1)/2)}{\Delta}\}dt\Big], \quad \zeta_0 = \Delta_0^{-R}. \tag{II.4.3}$$

From this we can read off the change-of-measure process which converts the reference measure $\mathbb{P}$ to the pricing measure $\tilde{\mathbb{P}}$, as well as the interest rate process. Indeed, these satisfy

$$dZ_t = -\frac{Z_t R\sigma}{\sqrt{\Delta}}dB_t, \quad \text{where } Z_t = \mathbb{E}[d\tilde{\mathbb{P}}/d\mathbb{P}|\mathcal{F}_t], \quad Z_0 = 1; \tag{II.4.4}$$

and

$$r_t = \rho - \beta R + \frac{R(A-\sigma^2(R+1)/2)}{\Delta}, \tag{II.4.5}$$

for $t \geq 0$. Under what conditions will the change-of-measure process Z actually be a martingale and not just a local martingale? The criterion is simple and complete:

Lemma II.4.1. *The process Z defined by (II.4.4) will be a martingale if and only*

[7] See Section II.8 for a slightly less restrictive scenario.

if

$$\frac{2A}{\sigma^2} \geq 2R + 1. \tag{II.4.6}$$

Proof. See Appendix A.1. ■

Notice that the spot rate process will be bounded below provided

$$A > \frac{\sigma^2(R+1)}{2}, \tag{II.4.7}$$

a condition implied by (II.4.6).

We will henceforth assume that inequality (II.4.6) holds. Now if we denote $\alpha = \rho - \beta R$ and $\gamma = R(A - (\sigma^2/2)(R+1))$ and write $r_t = \alpha + \frac{\gamma}{\Delta}$ from equation (II.4.5), then it is easy to see that in the measure $\tilde{\mathbb{P}}$, the spot rate process $(r_t)_{t\geq 0}$ satisfies the SDE

$$dr_t = (r_t - \alpha)\left\{\left[(\sigma^2(R+1) - A)\left(\frac{r_t - \alpha}{\gamma}\right) + \beta\right]dt - \sigma\sqrt{\frac{r_t - \alpha}{\gamma}}d\tilde{B}\right\} \tag{II.4.8}$$

where $\tilde{B}$ satisfies $d\tilde{B} = dB + (\sigma R/\sqrt{\Delta})dt$ and is a $\tilde{\mathbb{P}}$-Brownian Motion.

The SDE (II.4.8) becomes considerably neater for the case $\alpha = 0$ ($\rho = \beta R$); we then have

$$\frac{dr_t}{r_t} = \left[\frac{(\sigma^2(R+1) - A)}{\gamma}r_t + \beta\right]dt - \frac{\sigma}{\sqrt{\gamma}}\sqrt{r_t}d\tilde{B}. \tag{II.4.9}$$

We have determined (II.4.2) the candidate state-price density ζ, so from this and from the pricing relation (II.3.2) we expect[8] that

$$S_t^j \equiv S^j(\delta_t^j, \Delta_t) = \frac{1}{\zeta_t}\,\mathbb{E}\left[\int_t^\infty \delta_u^j \zeta_u \, du \,\middle|\, \mathcal{F}_t\right] \tag{II.4.10}$$

It remains to prove that what we suspect is an equilibrium for the economy actually is. To spell out what is required, we have to show that if we suppose that the price processes S^j are given by (II.4.10) and the spot rate process by (II.4.5), then the optimal consumption / investment policy for the representative agent whose wealth evolves as (II.3.1) is to take $\alpha^j(t) \equiv 1$ for all $t \geq 0$, for all $j = 1, \ldots, J$. The proof of this is a straightforward Lagrangian sufficiency

[8] The finiteness of the expectation in (II.4.10) is not immediately obvious; it turns out that the expectation is finite if $-R + 2A/\sigma^2 > 0$, which is implied by the assumed condition (II.4.6). See Lemma A.1.2 for more on this.

argument. Firstly, note from (II.4.10) that

$$\zeta_t S_t^j + \int_0^t \zeta_u \delta_u^j \, du \quad \text{is a martingale.} \tag{II.4.11}$$

(In fact, more is true. We shall prove in Lemma A.1.2 that $\zeta_0 S_0^j$ as defined in (II.4.10) is finite, and this implies by Doob's convergence theorem (see Rogers and Williams (2000, Th. II.69.2) that the process in (II.4.11) is a *uniformly integrable* martingale closed on the right by the random variable $\int_0^\infty \zeta_u \delta_u^j \, du$). In particular, the initial wealth X_0 of the representative agent who at time 0 holds all of the asset will be

$$X_0 = \zeta_0^{-1} \mathbb{E}\left[\int_0^\infty \zeta_u \Delta_u \, du \right] \tag{II.4.12}$$

Moreover, we see from (II.3.1) that the conjectured optimal policy of holding 1 unit of each of the assets for all time and consuming at rate Δ_t is indeed a feasible strategy, with corresponding non-negative wealth process $\Sigma \equiv (\Sigma_t := \sum_{j=1}^J S_t^j)_{t \geq 0}$.

For a general feasible pair (X, c), from (II.3.1) and Itô's Lemma we deduce that

$$\zeta_t X_t + \int_0^t \zeta_u c_u \, du \quad \text{is a non-negative local martingale,} \tag{II.4.13}$$

because the SDE for this process has no finite-variation term. A non-negative local martingale is a supermartingale, and therefore we have

$$\zeta_0 X_0 \geq \mathbb{E}\left[\int_0^\infty \zeta_u c_u \, du \right]. \tag{II.4.14}$$

Thus

$$\begin{aligned}
\mathbb{E}\big[\int_0^\infty U(t, c_t)\, dt\big] &\leq \mathbb{E}\big[\int_0^\infty \{U(t, c_t) - \zeta_t c_t\}\, dt\big] + \zeta_0 X_0 \\
&\leq \mathbb{E}\big[\int_0^\infty \{U(t, c_t^*) - \zeta_t c_t^*\}\, dt\big] + \zeta_0 X_0 \\
&= \mathbb{E}\big[\int_0^\infty U(t, c_t^*)\, dt\big].
\end{aligned}$$

The key point is the second line here, which follows precisely because $\zeta_t = U'(t, c_t^*) = U'(t, \Delta_t)$.

This establishes the claim that the conjectured equilibrium holds for this economy. In the following Sections, we shall proceed to apply the general pricing recipe (II.3.2) to compute prices of bonds, of the assets, and of options on the total assets for the particular model studied here.

II.5. Bessel Processes

We show in this Section that the solutions δ^j and Δ to the SDE's (II.2.1) and (II.2.2) are simple transformations of squared Bessel processes. This fact will then be used in the next Section to derive expressions for market prices of bonds and assets.

Let $W \equiv (W_t)_{t\geq 0}$ be a one-dimensional standard Brownian motion, let $\eta \geq 0$, and consider the SDE

$$dX_t = 2\sqrt{|X_t|}dW_t + \eta dt, \quad X_0 = x \geq 0. \tag{II.5.1}$$

The (pathwise unique) exact solution to (II.5.1) is called a squared Bessel process of dimension η started at x, and denoted by $BESQ^{\eta}(x)$. See Revuz and Yor (2001) for the basic properties of squared Bessel processes.

The parameter η is called the dimension of the process X. The transition density of X involves the Bessel function of index $\nu = \eta/2-1$; we shall write $BESQ^{(\nu)}(x)$ when it is more convenient to characterize the squared Bessel process by the index rather than dimension.

For later reference, we recall that the $BESQ^{\eta}(x)$ process has a transition density given by (see Revuz and Yor (2001))

$$\begin{aligned} q_t^{\eta}(x,y) &= \frac{1}{2t}\left(\frac{y}{x}\right)^{\nu/2} \exp\left(-(x+y)/2t\right) I_{\nu}(\sqrt{xy}/t) && \text{(II.5.2)} \\ &= \frac{1}{2t}e^{-(x+y)/2t} \sum_{k\geq 0} \frac{(\frac{x}{2t})^k (\frac{y}{2t})^{k+\nu}}{k!\,\Gamma(k+\nu+1)} && \text{(II.5.3)} \end{aligned}$$

valid for non-negative x and y, and for $t > 0$. Here, $I_{\nu}(\cdot)$ is the modified Bessel function of the first kind of index ν, where ν is related to η by

$$\nu = \eta/2 - 1.$$

The Laplace transform of X_t is

$$\mathbb{E}^x[\exp(-\lambda X_t)] \equiv \phi^\eta(\lambda, t, x) = (1+2\lambda t)^{-\eta/2}\exp(-\lambda x/(1+2\lambda t)),$$
$$\lambda > 0,\ t > 0,\ x > 0. \qquad \text{(II.5.4)}$$

We now give two results, involving expectations of functions of squared Bessel processes, that are the basis for our calculations in Section II.6.

Lemma II.5.1. *Let $X \equiv (X_t)_{t\geq 0}$ be a $BESQ^\eta(x)$ process, denote its law by $\mathbb{P}^x$, and let $R > 0$. If the condition*

$$\nu + 1 - R > 0 \qquad \text{(II.5.5)}$$

holds, then

$$\mathbb{E}^x(X_t^{-R}) = \frac{\Gamma(\nu+1-R)}{\Gamma(\nu+1)}\exp(-x/2t)(2t)^{-R}\ {}_1F_1(\nu+1-R, \nu+1, x/2t), \qquad \text{(II.5.6)}$$

where $x \geq 0$, $\nu = \eta/2 - 1$ and where the function ${}_1F_1(\cdot,\cdot,\cdot)$ is the confluent hypergeometric function defined by

$${}_1F_1(a,b,z) = \sum_{j=0}^{\infty}\frac{\Gamma(a+j)}{\Gamma(a)}\frac{\Gamma(b)}{\Gamma(b+j)}\frac{z^j}{j!}. \qquad \text{(II.5.7)}$$

Proof. Consider $x > 0$ first. Let $q_t^\eta(x,\cdot)$ be the transition density of $BESQ^\eta(x)$ given in equation (II.5.2). Then

$$\begin{aligned}
\mathbb{E}^x(X_t^{-R}) &= \int_0^\infty y^{-R} q_t^\eta(x,y)dy \\
&= \int_0^\infty y^{-R}(y/x)^{\nu/2}(2t)^{-1}\exp(-(x+y)/2t)I_\nu(\sqrt{xy}/t)dy \\
&= \int_0^\infty y^{-R}(y/x)^{\nu/2}(2t)^{-1}\exp(-(x+y)/2t)\sum_{j=0}^{\infty}\frac{\left(\sqrt{xy}/2t\right)^{2j+\nu}}{j!\,\Gamma(j+\nu+1)}dy.
\end{aligned}$$

The integrand here is non-negative; after interchanging the order of summation and integration and making the substitution $z = y/2t$, we recognize part of the integrand as a gamma density. Because of the condition (II.5.5), we can integrate this out, which leaves us with

$$\mathbb{E}^x(X_t^{-R}) = \exp(-x/2t)(2t)^{-R}\sum_{j=0}^{\infty}(x/2t)^j\frac{\Gamma(\nu+j+1-R)}{j!\,\Gamma(\nu+j+1)}$$
$$= \frac{\Gamma(\nu+1-R)}{\Gamma(\nu+1)}\exp(-x/2t)(2t)^{-R}\ {}_1F_1(\nu+1-R,\nu+1,x/2t),$$

as we want.

The case $x=0$ is similar but easier, so we omit details. ■

In the next Lemma we use the explicit form (II.5.4) for the Laplace transform of the squared Bessel transition density to compute the expectation of a function of two independent squared Bessel Processes. This computation can be used to derive expressions for prices of the single assets.

Lemma II.5.2. *Let $X \equiv (X_t)_{t\geq 0}$ (resp. $Y \equiv (Y_t)_{t\geq 0}$) be $BESQ^\eta(x)$ (resp. $BESQ^\mu(y)$), two independent squared Bessel processes, let $\mathbb{P}^{(x,y)}$ denote their joint law, and let $R>0$. Then*

$$\mathbb{E}^{(x,y)}\left[Y_t(X_t+Y_t)^{-R}\right] = -\frac{\partial}{\partial\theta}\left[\int_0^\infty \phi^\eta(\lambda,t,x)\phi^\mu(\lambda+\theta,t,y)\frac{\lambda^{R-1}}{\Gamma(R)}d\lambda\right]\Bigg|_{\theta=0}$$
$$= \int_0^\infty (1+2\lambda t)^{-\frac{\eta+\mu}{2}-1}e^{-\frac{\lambda(x+y)}{1+2\lambda t}}\left(\mu t+\frac{y}{1+2\lambda t}\right)\frac{\lambda^{R-1}}{\Gamma(R)}d\lambda. \tag{II.5.8}$$

This integral is finite if

$$\frac{\eta+\mu}{2}+1-R>0. \tag{II.5.9}$$

Proof. For $\lambda>0$, $\theta>0$, consider the joint Laplace transform

$$\mathbb{E}^{(x,y)}\left[\exp(-\lambda(X_t+Y_t)-\theta Y_t)\right] = \phi^\eta(\lambda,t,x)\phi^\mu(\lambda+\theta,t,y). \tag{II.5.10}$$

Multiplying both sides of this equation by $\lambda^{R-1}/\Gamma(R)$ and integrating with respect to λ, we get

$$\int_0^\infty \mathbb{E}^{(x,y)}\exp(-\theta Y_t)\exp(-\lambda(X_t+Y_t))\lambda^{R-1}\frac{d\lambda}{\Gamma(R)}$$
$$= \int_0^\infty \phi^\eta(\lambda,t,x)\phi^\mu(\lambda+\theta,t,y)\lambda^{R-1}\frac{d\lambda}{\Gamma(R)}. \tag{II.5.11}$$

On the left hand side, changing the order of integration transforms the expression to

$$\mathbb{E}^{(x,y)}\big[\exp(-\theta Y_t)(X_t+Y_t)^{-R}\big]. \tag{II.5.12}$$

Differentiating this expression, and the right side of (II.5.11), with respect to θ, using the Laplace transform given in (II.5.4), gives (II.5.8).

Let us now verify the integrability condition (II.5.9). In equation (II.5.8), make the substitution $\alpha = (1+2\lambda t)^{-1}$, to get

$$\mathbb{E}^{(x,y)}\big[Y_t(X_t+Y_t)^{-R}\big] = \exp\big(-(x+y)/2t\big)\frac{1}{(2t)^R\Gamma(R)} \\ \times \int_0^1 \alpha^{(\eta+\mu)/2-1}\exp\big((x+y)\alpha/2t\big)(\mu t + y\alpha)(1/\alpha - 1)^{R-1}d\alpha. \tag{II.5.13}$$

For small α, the integrand is proportional to

$$\alpha^{(\eta+\mu)/2-R}\exp\big(\alpha(x+y)/2t\big)(\mu t + y\alpha),$$

which shows why we need the condition (II.5.9). ■

We now exhibit the solution Δ to the SDE (II.2.2) as a transformation of a squared Bessel process. Thus, let $(\Delta_t)_{t\geq 0}$ be a solution to (II.2.2). A simple Itô calculation verifies that the process defined by $\tilde{Y}_t = \exp(\beta t)\Delta_t$ satisfies the SDE

$$d\tilde{Y}_t = \exp(\beta t/2)\sigma\sqrt{\tilde{Y}_t}dB_t + A\exp(\beta t)dt,$$

with $\tilde{Y}_0 = \Delta_0$. This says that for $f \in C^2$, the process defined by

$$\tilde{M}_t = f(\tilde{Y}_t) - f(\tilde{Y}_0) - \int_0^t \tilde{\mathcal{G}} f_s ds$$

is a martingale, where $\tilde{\mathcal{G}} f_t \equiv \exp(\beta t)\big[(\sigma^2/2)\tilde{Y}_t f''(\tilde{Y}_t) + Af'(\tilde{Y}_t)\big]$. If we now change the time scale via the deterministic clock $A_t \equiv \int_0^t \lambda\exp(\beta s)ds = (\lambda/\beta)(e^{\beta t}-1)$ with continuous inverse $\tau_t = \inf\{u : A_u > t\}$, so that $Y_t \equiv \tilde{Y}_{\tau_t}$, then $\tilde{M}$ time-changes to the martingale

$$M_t \equiv \tilde{M}_{\tau_t} = f(Y_t) - f(Y_0) - \int_0^t \mathcal{G} f(Y_s)ds,$$

where $\mathcal{G}f(y) = (\sigma^2/2\lambda)yf''(y) + (A/\lambda)f'(y)$. Thus, the process Y satisfies the SDE

$$dY_t = \frac{\sigma}{\sqrt{\lambda}}\sqrt{Y_t}dB_t + (A/\lambda)dt. \tag{II.5.14}$$

Choosing

$$\lambda = \sigma^2/4,$$

we recognise (II.5.14) as the $BESQ^\eta$ SDE, with

$$\eta = 4A/\sigma^2.$$

Thus, we have the following result.

Lemma II.5.3. *The process Δ satisfying SDE (II.2.2) can be written as*

$$\Delta_t = \exp(-\beta t)\tilde{Y}_t = \exp(-\beta t)Y_{A_t}, \tag{II.5.15}$$

where $A_t = (\lambda/\beta)(e^{\beta t} - 1)$, with $\lambda = \sigma^2/4$ and where Y is $BESQ^\eta(\Delta_0)$, with $\eta = 4A/\sigma^2$.

Obviously, an analogous result holds for the processes δ^j, $1 \leq j \leq J$. Now we are able to read off the distribution of the process Δ. Indeed, using the characterisation in Lemma II.5.3 and recalling the form of the Laplace transform of the squared Bessel process given by expression II.5.4, we obtain the Laplace transform of Δ as

$$\mathbb{E}^x\left[\exp\left(-\alpha\Delta_t\right)\right] = \left(1 + 2A_t\alpha e^{-\beta t}\right)^{-\eta/2} \exp\left(\frac{-x\alpha e^{-\beta t}}{1 + 2A_t\alpha e^{-\beta t}}\right). \tag{II.5.16}$$

This characterizes the law of Δ_t as that of the random variable $e^{-\beta t}A_t\chi^2(\eta, x/A_t)$, where $\chi^2(a,b)$ has the non-central χ^2 distribution of degrees of freedom a and non-centrality b. This law is known to appear in connection with the CIR process; in particular, if we let $t \to \infty$ in (II.5.16) we harvest the result, proved by alternative means in Appendix A.1, that the invariant law of Δ is a Gamma distribution.

II.6. Bond and asset prices

In this Section, we apply the various results just established to derive expressions for bond prices, for the price of the total assets in the market, and for options on the assets. These are not of course as explicit as the prices in the Black-Scholes model, but they are perfectly tractable numerically.

II.6.1. Bond prices

The price $P(0,T)$ of the zero-coupon bond maturing at T is

$$\begin{aligned} P(0,T) &= \frac{1}{\zeta_0}\mathbb{E}[\zeta_T|\mathcal{F}_0] \\ &= e^{-\rho T}z^R\mathbb{E}^z[\Delta_T^{-R}], \end{aligned} \tag{II.6.1}$$

where $z \equiv \Delta_0$. But from Lemma II.5.3, we can compute the expectation above as

$$\begin{aligned} P(0,T) &= e^{-\rho T}z^R\mathbb{E}^z[e^{\beta RT}Y_{A_T}^{-R}] \\ &= z^Re^{-\rho T}e^{\beta RT}e^{-z/2A_T}(2A_T)^{-R}\frac{\Gamma(\nu+1-R)}{\Gamma(\nu+1)}\,{}_1F_1(\nu+1-R,\nu+1,z/2A_T), \end{aligned} \tag{II.6.2}$$

where we have used (II.5.15) to write Δ in terms of the $BESQ^\eta(z)$ process Y, with the clock A_t being as in Lemma II.5.1, $\eta = 4A/\sigma^2$, $\nu = \eta/2 - 1$.

II.6.2. Bond Yield Volatilities

Notice from Equation (II.6.2) that the price now of a bond maturing in T time units from now is a function of the level of Δ now only (apart from the model parameters, of course). If we take $z = \Delta_t$ in Equation (II.6.2), then $P(0,T) \equiv g(z)$ is the price of a bond written at t with lifetime T. Therefore, the SDE for the yield of such a bond is

$$\begin{aligned} d(-\log g(\Delta_t)/T) = -\frac{1}{Tg(\Delta_t)}\Bigg[&\left(g'(\Delta_t)(A-\beta\Delta)+\frac{1}{2}g''(\Delta_t)\sigma^2\Delta_t\right)dt \\ &+\left(g'(\Delta_t)\sigma\sqrt{\Delta_t}dB_t\right)\Bigg]. \end{aligned} \tag{II.6.3}$$

By computing bond prices for different levels z, we can estimate numerically the values of the volatility coefficient

$$\frac{g'(z)}{Tg(z)}\sigma\sqrt{z} \tag{II.6.4}$$

in the above SDE. This is done in Section II.7.4.

II.6.3. The Asset price processes

Since the processes δ^j are independent, we can write $\Delta \equiv (\bar{\Delta}^j + \delta^j)$ as the sum of two independent squared Bessel processes. Specifically, we represent

$$\delta^j(s) = \exp(-\beta s)Y_{A_s}, \quad \bar{\Delta}^j(s) = \exp(-\beta s)X_{A_s}$$

where

$$X \text{ is } BESQ^{\eta}(x), \quad \eta = 4(A - a_j)/\sigma^2,$$

and

$$Y \text{ is } BESQ^{\mu}(y), \quad \mu = 4a_j/\sigma^2$$

and $y = \delta_0^j$, $x = \Delta_0 - \delta_0^j$. Using the pricing relation (II.3.7) and the representation of Lemma II.5.3,

$$S_0^j\zeta_0 = S_0^j(x+y)^{-R} = \int_0^\infty \mathbb{E}^{(x,y)}\left[e^{-\rho s}e^{(R-1)\beta s}Y_{A_s}(Y_{A_s} + X_{A_s})^{-R}\right]ds, \tag{II.6.5}$$

where $\mathbb{P}^{(x,y)}$ denotes the joint law of the process (X, Y) started from $(X_0, Y_0) = (x, y)$. Using Lemma II.5.2, we obtain an expression for the price of the j'th asset:

$$\begin{aligned} S_0^j = (x+y)^R \int_0^\infty & e^{((R-1)\beta-\rho)s} \\ \times & \left\{ \int_0^\infty (1+2\lambda A_s)^{-\frac{\mu+\eta}{2}-1} e^{-\frac{\lambda(x+y)}{1+2\lambda A_s}} \left(\mu A_s + \frac{y}{1+2\lambda A_s}\right) \frac{\lambda^{R-1}}{\Gamma(R)} d\lambda \right\} ds. \end{aligned} \tag{II.6.6}$$

This expression can be and has to be computed numerically, but in the sequel we focus attention on the sum of the asset prices $\sum_{j=1}^J S^j$.

Writing $\Sigma = (\Sigma_t)_{t\geq 0}$ for the price process of the total assets, the pricing relation (II.3.7) gives us

$$\begin{aligned} \zeta_0\Sigma_0 &= \mathbb{E}\Big[\int_0^\infty \Delta_u\zeta_u du\Big] \\ &= \int_0^\infty e^{-\rho u}\mathbb{E}^x(\Delta_u^{1-R})du, \end{aligned} \tag{II.6.7}$$

with $x = \Delta_0$. From this, using again the representation of Lemma II.5.3 and the computation in Lemma II.5.1, we get

$$\begin{aligned} \Sigma_t = z^R \int_0^\infty & e^{-\rho u}e^{\beta(R-1)u}\exp(-z/2A_u)(2A_u)^{1-R} \\ &\times \frac{\Gamma(\nu+2-R)}{\Gamma(\nu+1)}\,{}_1F_1(\nu+2-R,\nu+1,z/2A_u)du, \end{aligned} \tag{II.6.8}$$

where now $z = \Delta_t$. The change of variable $s = 2A_u$ followed by $v = z/s$ gives an equivalent expression for Σ_t:

$$\Sigma_t = f(z) \equiv z^2\frac{1}{2\lambda}\int_0^\infty\Big(1+\frac{\beta}{2\lambda}\frac{z}{v}\Big)^{-\theta}e^{-v}v^{R-3}\frac{\Gamma(c)}{\Gamma(d)}\,{}_1F_1(c,d,v)dv, \tag{II.6.9}$$

where $c \equiv \nu+2-R$, $d \equiv \nu+1$, and $\theta \equiv 2+(\rho/\beta)-R$.

The finiteness of these integrals is guaranteed (see Lemma A.1.2) if we have

$$\nu + 2 - R > 0. \tag{II.6.10}$$

From (II.6.8), we see that the time-t price Σ_t is a function of only Δ_t. Thus, $\Sigma_t = f(\Delta_t)$ where the function f is defined by the right side of (II.6.8) or of (II.6.9) and can be computed numerically. Even more, Σ is a diffusion satisfying the SDE

$$\begin{aligned} d\Sigma_t &= \{f'(\Delta_t)(A-\beta\Delta)+\frac{1}{2}f''(\Delta_t)\sigma^2\Delta\}dt + f'(\Delta_t)\sigma\sqrt{\Delta_t}dB_t \\ &=: a(\Sigma_t)dt + b(\Sigma_t)dB_t, \end{aligned} \tag{II.6.11}$$

where we can in principle write Δ in terms of Σ as $\Delta_t = f^{-1}(\Sigma_t)$. Numerical estimation of derivatives of f by finite differencing allows us to characterize the diffusion Σ, and details of this are given in Section II.7.3.

II.6.4. Asymptotic behaviour of f near zero

Consider again the expression (II.6.9) (which we recall is finite for all $z > 0$ as long as the condition (II.6.10) holds):

$$\begin{aligned} f(z) &= \frac{z^2}{2\lambda} \int_0^\infty \left(1 + \frac{\beta}{2\lambda}\frac{z}{v}\right)^{-\theta} e^{-v} v^{R-3} \frac{\Gamma(c)}{\Gamma(d)}\, {}_1F_1(c,d,v)dv \\ &= \frac{z^2}{2\lambda} \sum_{k\geq 0} \int_0^\infty \left(1 + \frac{\beta}{2\lambda}\frac{z}{v}\right)^{-\theta} e^{-v} v^{R+k-3} \frac{\Gamma(c+k)}{k!\,\Gamma(d+k)} dv, \qquad \text{(II.6.12)} \\ &\equiv \frac{z^2}{2\lambda} \sum_{k\geq 0} \frac{\Gamma(c+k)}{k!\,\Gamma(d+k)} f_k(z) \end{aligned}$$

say, where of course

$$f_k(z) \equiv \int_0^\infty \left(1 + \frac{\beta z}{2\lambda v}\right)^{-\theta} e^{-v} v^{R+k-3} dv.$$

We shall determine the asymptotics of $f(z)$ as $z \downarrow 0$ for different parameter regimes[9]. By monotone convergence, the limit as $z \downarrow 0$ of $f_k(z)$ is finite if and only if $R + k - 2 > 0$, and the limit value is then $\Gamma(R + k - 2)$. We therefore have a complete resolution of the following cases.

Case 1: $2 < R < \nu + 2$[10]. For this case we obtain[11]

$$f(z) \sim \frac{z^2}{2\lambda} \sum_{k\geq 0} \frac{\Gamma(c+k)\Gamma(R+k-2)}{\Gamma(k+1)\Gamma(d+k)} \qquad (z \downarrow 0).$$

Stirling's formula shows that for large k, the sum has terms decaying as k^{-2} and is therefore convergent.

Case 2: $R = 2 < \nu + 2$. In this case, the terms $f_k(z)$ are convergent to finite limits except for $k = 0$, where we have

$$f_0(z) = \int_0^\infty \left(1 + \frac{\beta z}{2\lambda v}\right)^{-\theta} e^{-v} \frac{dv}{v} \sim \log(1/z) \quad (z \downarrow 0).$$

[9] Thanks to Alexander Cherny (personal communication) who produced a first proof of these asymptotics, using methods of sample-path estimation.

[10] The inequality $R < \nu + 2$ is simply (II.6.10).

[11] For two functions h and g, we write $h \sim g$ to mean that $\lim(h(z)/g(z)) = 1$.

To see this, note that

$$\begin{aligned} f_0'(z) &= -\frac{\theta}{z}\int_0^\infty (1+u^{-1})^{-\theta-1}u^{-2}e^{-\beta zu/2\lambda}\,du, \\ &\sim -\frac{1}{z} \quad (z \downarrow 0) \end{aligned}$$

where the integral converges (to $1/\theta$) as $z \downarrow 0$ because $\theta = \rho/\beta > 0$.

Assembling, we get that

$$f(z) \sim \frac{1}{2\lambda}\, z^2 \log(1/z) \quad (z \downarrow 0)$$

in this case.

Case 3: $R < \min\{2, \nu + 2\}$. Once again, it is the term $f_0(z)$ which dominates, and if we write $\varepsilon = 2 - R > 0$, by change of variables in the integral we have

$$\begin{aligned} f_0(z) &= \int_0^\infty \left(1+\frac{\beta}{2\lambda u}\right)^{-\theta} (zu)^{-1-\varepsilon}e^{-zu}z du \\ &\sim z^{-\varepsilon}\int_0^\infty \left(1+\frac{\beta}{2\lambda u}\right)^{-\theta} u^{-1-\varepsilon}du \quad (z \downarrow 0), \end{aligned}$$

the final integral being convergent because $\theta = \varepsilon + \rho/\beta$. We deduce the asymptotics:

$$f(z) \sim Cz^R \quad (z \downarrow 0).$$

In practice, the risk aversion coefficient $R > 2$, so that we expect the asset price to be quadratic in Δ when Δ is close to zero. See Section II.7.3 for numerical evidence supporting this.

II.6.5. A closed-form expression for the function f

If with $R > 2$ we set $\theta = 0$, equivalent to the special choice of parameter

$$\rho = \beta(R-2),$$

the asymptotic form derived above for Case 1 ($2 < R < \nu + 2$) is in fact valid for all values $z \geq 0$. Thus,

$$\begin{aligned} f(z) &= \frac{z^2}{2\lambda} \sum_{k \geq 0} \frac{\Gamma(c+k)\Gamma(R+k-2)}{\Gamma(k+1)\Gamma(d+k)} \\ &= \frac{z^2}{2\lambda} \frac{\Gamma(c)\Gamma(R-2)}{\Gamma(d)} \, {}_2F_1(R-2, c; d; 1) \quad \forall z \geq 0, \end{aligned}$$

where

$${}_2F_1(a, b; k; x) = \frac{\Gamma(k)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(k+n)} \frac{x^n}{n!}$$

is the Gauss hypergeometric series discussed in, for example, Ch. 15 of Abramowitz and Stegun (1964).

If we denote

$$\gamma := \frac{1}{2\lambda} \frac{\Gamma(c)\Gamma(R-2)}{\Gamma(d)} \, {}_2F_1(R-2, c; d; 1),$$

then $f(z) = \gamma z^2$ and the SDE (II.6.11) specializes to

$$d\Sigma_t = 2\gamma^{1/4}\sigma\Sigma_t^{3/4} dB_t + \gamma[(\sigma^2 + 2A)\sqrt{\Sigma_t/\gamma} - 2(\beta/\gamma)\Sigma_t]dt. \qquad \text{(II.6.13)}$$

In Section II.7.3, we compare this analytical result with numerical estimates.

II.6.6. Prices of options

Given that we can compute the total asset price, the price of an option on the total asset price is only an integration away, because the transition density of the underlying process Δ is known to us by virtue of the Lemma II.5.3. So consider a European call option on the total asset price S written at time $t = 0$ (when the value of Δ is Δ_0) with expiry date T and strike price K. The time-0 value of the option is then

$$\frac{1}{\zeta_0} \mathbb{E}\left[\zeta_T (\Sigma_T - K)^+ \middle| \mathcal{F}_0 \right].$$

Writing $\Sigma_T = f(\Delta_T)$ explicitly in terms of the underlying process Δ, and recalling the expression (II.4.2) for the state price density, the above expectation becomes

$$
\begin{aligned}
&x^R e^{-\rho T}\mathbb{E}^x\left[\Delta_T^{-R}(f(\Delta_T)-K)^+\right]\\
&= e^{(R\beta-\rho)T}x^R\mathbb{E}^x\left[Y(A_T)^{-R}\big(f(e^{-\beta T}Y(A_T))-K\big)^+\right]\\
&= e^{(R\beta-\rho)T}x^R\int_0^\infty y^{-R}\Big(f(ye^{-\beta T})-K\Big)^+ q_{A(T)}(x,y)dy
\end{aligned}
\qquad \text{(II.6.14)}
$$

where, with $x \equiv \Delta_0$ and $\nu = 2A/\sigma^2 - 1$, Lemma (II.5.3) has been used to write Δ in terms of the $BESQ^{(\nu)}(x)$ process Y whose transition density $q_t(x,\cdot)$ is given from (II.5.2). We discuss evaluation of option prices in Section II.7.4.

II.7. Calibration and Numerical Results

We now discuss the numerical evaluation of the various pricing expressions derived in the previous Sections, and also explain how typical parameter values for the model were obtained by using bond price data.

The model is parametrized by A, σ, β, R, and ρ. We recall that A and β control the mean reversion, and σ the volatility, of the ergodic diffusion Δ. R is the coefficient of relative risk aversion, assumed constant, and ρ is the discount factor for the utility of the agent. Apart from these parameters, the time-t prices of bonds and of the total asset as given in expressions (II.6.2) and (II.6.8) depend only on the value $z = \Delta_t$.

II.7.1. Numerical evaluation of model prices for bonds

Given model parameters and a time-t level $z = \Delta_t$ for Δ, computing time-t bond prices from equation (II.6.2) involves evaluating the hypergeometric function ${}_1F_1(a,b,x)$ for arguments $a = \nu + 1 - R = (2A/\sigma^2) - R$, $b = \nu + 1 = 2A/\sigma^2$ and $x = z/2A_T$. Because $R > 0$ and because of the condition (II.5.5), we shall have $b > a > 0$; the function ${}_1F_1(a,b,x)$ is then well-defined for all values of x that are contingent on the maturity date T.

Unfortunately, no general method exists to evaluate the hypergeometric function for a wide range of argument values. In our computations, the argument x can become very large when the maturity time T is small, and this is problematic unless the evaluation procedure is chosen properly.

Muller (2001) advocates choosing a method depending on the values of $R_1 \equiv ax/b$ and $R_2 \equiv a(b-a)/x$, and we found this to work well with some modifications. His Method 1 involves adding a finite number of terms of the series defining ${}_1F_1(a,b,x)$. We use this method when $R_1 < 30$.

If $R_2 < 1$ and $x > 400$, we use an asymptotic series in x^{-1}, given in formula 13.5.1 of Abramowitz and Stegun (1964). This is also Method 2 of Muller, who suggests using it if $R_2 < 1$ and $x > 50$. In simulation tests, we found that this cutoff value of 50 for x is not large enough, which is why we increased it to 400.

The most reliable method seems to be the rational approximation of Luke, referred to as Method 3 in Muller (2001). We implemented this method using the freely available[12] SAS code of Muller adapted to the Scilab environment, and used it whenever the criteria for x, R_1 and R_2 described above were not satisfied.

II.7.2. Calibration Procedure

In order to calibrate the model, we searched for optimal parameter vectors $\theta = (A, \sigma, \beta, R, z, \rho)$ such that the prices for bonds, as computed from the expression (II.6.2) are close, in some appropriate sense, to actually observed bond prices. The level z of Δ has therefore been treated as one of the parameters in the fitting procedure.

The data consisted of a time series of $N + 1 = 278$ daily consecutive prices for zero coupon bonds on the US dollar, for maturities which we shall denote by the vector $M = (1/12, 1/4, 1/2, 1, 2, 5, 7, 10)$ whose entries are in years. We shall denote the data by (Y_i^n), where for $0 \le n \le N$ and $1 \le i \le 8$, Y_i^n represents the price observed on the n'th day for the bond maturing in $M(i)$ years from that day. The corresponding prices obtained from the model equation (II.6.2) shall be denoted by $(P_i^n(\theta))$.

We adopted as error-of-fit criterion the function

$$MAD_n(\theta) := \frac{1}{8}\sum_{i=1}^{8} \left|\frac{Y_i^n - P_i^n(\theta)}{Y_i^n}\right|, \tag{II.7.1}$$

[12] http://www.bios.unc.edu/~muller.

defined for each day n of data, $0 \leq n \leq N$.

As a first attempt at calibration, we took the function (II.7.1) as our objective and estimated a time series of parameters $(\hat{\theta}_n)$ such that for each n, $\hat{\theta}_n$ minimizes $MAD_n(\cdot)$ subject to the parameter conditions (II.5.5) and (A.1.3). On each day, the starting iterate for the minimization was chosen to be some fixed vector θ which gives a reasonable fit to typical values in our data set.

Although reasonably good fits were obtained the parameter time series $(\hat{\theta}_n)$ was not as stable as one would wish.[13] To remedy this, we redid the minimization procedure using the cost function

$$MADV_n(\theta) := MAD_n(\theta) + |\theta - \hat{\theta}_{n-1}|^2, \tag{II.7.2}$$

which penalizes day-to-day variation in the parameter vector θ.

The form of the penalty term here is not the most natural one to choose. In particular, because typical parameter values are of different orders of magnitude, the weightings for parameter variation implied in the cost function (II.7.2) are unequal. The results of using an equally weighted cost function, (by penalizing changes in parameter values *relative* to values that are the result of the previous day's optimization, say) were qualitatively similar, and in general did not improve quality of fit. We also tried a likelihood approach, as follows. We suppose that rather than observing true bond prices we observe bond prices plus an independent noise term, and also suppose that parameters are conditionally Gaussian. The process Δ is allowed to vary from day to day according to its (known) transition density, and parameter fits are obtained by maximizing an appropriate likelihood. This approach is more flexible and intuitive, because weights can now be interpreted as variances of error terms. However, we found that quality of fit and parameter stability were inferior to those obtained with cost function (II.7.2).

In Table II.1 we report the descriptive statistics for quality of fit (expressed through the criterion $MAD(\cdot)$) resulting from using $MAD(\cdot)$ (left column) and $MADV(\cdot)$ (right column) as cost functions. As expected, a slight loss of quality of fit results from using $MADV(\cdot)$ but this is a small price to pay for the appreciable gain in parameter stability. We see in the first two columns of Table II.2 that parameter variance is reduced by several orders of magnitude in some cases[14]

[13] If the model is correct, then we logically expect the time series $(\hat{\theta}_n)$ to stay constant from one day to the next.

[14] especially for R and A which are large compared to remaining model parameters and are

Table II.1: Description of $MAD(\cdot)$ errors-of-fit for the two cost funtions.

	MAD	MADV
	(basis points)	
Min.	4.092	4.670
1st Qu.	7.301	7.899
Median	9.297	10.076
Mean	10.589	10.726
3rd Qu.	12.982	12.507
Max.	22.707	23.422

when using $MADV(\cdot)$ as opposed to $MAD(\cdot)$ as objective function. In the right column of the same table, we present values of parameters on the day of best fit for one of several calibration runs that we performed. This parameter set was used for all numerical studies of asset prices presented below.

II.7.3. Total Asset Price

Fixing model parameters A, σ, β, R, and ρ, the time-t total asset price $\Sigma_t = f(\Delta_t)$, as discussed in Section II.6.3. Here, the function f defining Σ in terms of Δ is the right hand side of the expression (II.6.8) and can be evaluated numerically.

We set model parameters as in the rightmost column of Table II.2. The parameter z, which we recall is the time-t level of Δ, is now considered to be a variable, rather than a fitted value, in order that we can study the dependence of Σ on Δ. We evaluated $f(z)$ on a sparse grid spanning z-values from 0 up to three standard deviations beyond the mean of the stationary law of Δ. Interpolation was then used to obtain values of f on a much finer grid.

The resulting prices are plotted in the first frame in Figure II.1. and on a log-log scale in the second frame. The remaining two plots exhibit the form of the volatility and drift coefficients $b(\cdot)$ and $a(\cdot)$, respectively, in the SDE (II.6.11). The slope of the log-log plot for the volatility coefficient $b(\cdot)$ is estimated at

thus more heavily penalized by the choice of penalty term in Equation (II.7.2).

Table II.2: Variances of parameter estimates resulting from minimizing the two cost functions, and best values obtained from minimizing $MADV(\cdot)$ cost function.

	MAD	MADV	'Best' values
	(units of 10^{-6})		
A	455.1479	9.4801	0.5189612
σ	126.0241	46.3808	0.1483002
β	86.0588	4.5066	0.207032
R	139.6877	0.1777	3.04367
ρ	10.7768	8.0131	0.078836

0.8065, and this is close to the value 0.75 which is valid asymptotically as $\Delta \to 0$ (cf. Section II.6.4).

For the special case $\rho = \beta(R-2)$, the function f is known exactly in closed form, as explained in Section II.6.5. Figure II.2 shows the same characteristics for the diffusion Σ, evaluated numerically, for parameter choices as those described above, but with $\rho = \beta(R-2)$. The estimate of the form of the volatility is as expected, the exponent 0.75 now being valid for all values of Δ. The constant γ defined in Section II.6.5 is estimated (from the log-linear relationship in the third frame) as 1.93, which is to be compared to its analytical value 1.92987. This example would lead to a value of $\rho = \beta(R-2) = 21.59\%$, an unrealistically high value; we include it merely to confirm the conclusions of Section II.6.5.

Figure II.3 shows how total asset price varies as a function of σ, the volatility coefficient of the aggregate rate of output, and the risk aversion R. The surface seen is the result of the interplay between what are usually referred to as the income and substitution effects in economics (we refer the reader to a standard textbook such as Burda and Wyplosz (1997) for an accessible description).

It is generally true (see, for example, Basak and Cuoco (1998)) that increasing consumption volatility σ causes the spot rate r_t to decrease and the value of the cashflow Δ to *in*crease. The former effect is due to the precautionary savings motive, (the *prudence* coefficient is positive for our example), the latter to an increase in the risk premium. On the other hand, if risk aversion R increases,

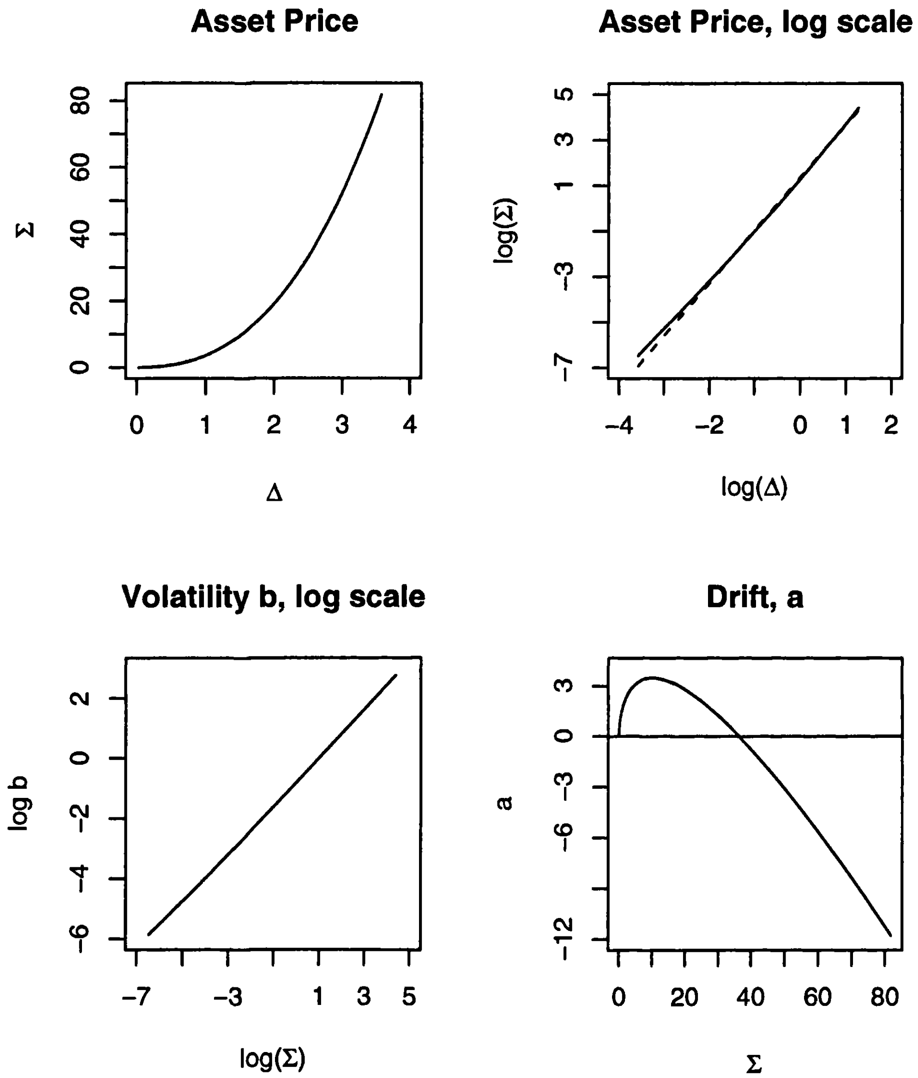


Figure II.1: Characteristics of the diffusion Σ. Model parameters are $A = 0.5189612$, $\sigma = 0.1483002$, $\beta = 0.207032$, $R = 3.04367$, $\rho = 0.07883598$. Estimated Regression model for asset price, shown as dotted line, is $f(z) = 3.9041\ z^{2.3169}$. Estimated volatility is $b(s) = 0.4566\ s^{0.8065}$.

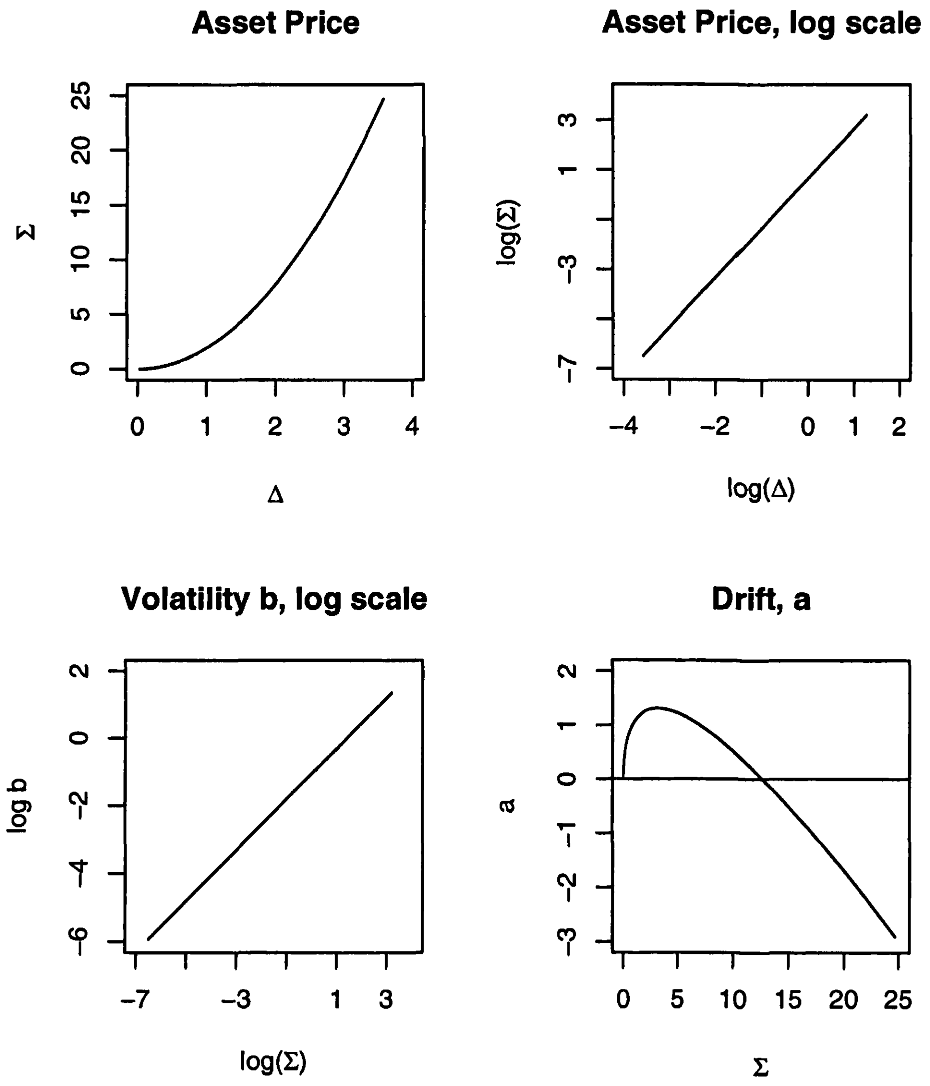


Figure II.2: Characteristics of the diffusion Σ. Model parameters are $A = 0.5189612$, $\sigma = 0.1483002$, $\beta = 0.207032$, $R = 3.04367$, $\rho = \beta(R-2)$. Estimated Regression model agrees perfectly with analytical form; $f(z) = 1.93\ z^2$, $b(s) = 0.5724\ s^{0.75}$.

demand for the risky asset falls, so that its equilibrium price drops in tandem. However, changes in price induce substitution effects on demand, which affect the equilibrium and result in what we see in Figure II.3.

For a risk aversion $R = 1$, when the agent has logarithmic utility, we see that asset price is inelastic with respect to σ. Mathematically, it is quite obvious why this happens; for $R = 1$ in our example, the asset price at time t satisfies

$$x^{-1}\Sigma_t = \mathbb{E}^x \int_0^\infty e^{-\rho t} dt = 1/\rho, \tag{II.7.3}$$

where $x \equiv \Delta_t$. Figure II.3, computed with $x = 2.5$, is consistent with the value that we expect from Equation (II.7.3). This inelasticity effect happens more generally in economics, where it is well known that for logarithmic utility, the income and substitution effects offset exactly.

For $R < 1$, we notice that the asset price rises with σ. This implies that larger uncertainty in consumption leads to *higher* consumption, which in turn means that the 'income' effect dominates. On the other hand, it is the 'substitution' effect which is the stronger for $R > 1$, so that there is a 'flight to bonds' as volatility rises in this regime of higher risk aversion.

II.7.4. Option and Bond Yield Volatilities

In Section II.6.6 we exhibited the value of a (European) option as an integral with respect to the density function of a certain squared Bessel process. Computationally, evaluating the integral involves the integration of the modified Bessel function of order ν.

For our choice of parameters, the index ν lies in the region of 40, and for small values of the maturity time T, we are dealing here with evaluating the modified Bessel function at arguments of the order of $10^3 - 10^4$, where the function available in Scilab fails. To get around this problem, we used the relation, given in formula 13.6.3 of Abramowitz and Stegun (1964), between the modified Bessel function and the confluent hypergeometric function:

$$e^{-z} I_\nu(z) = (z/2)^\nu \frac{1}{\Gamma(\nu+1)} e^{-2z} \ {}_1F_1(\nu + 1/2, 2\nu + 1, 2z). \tag{II.7.4}$$

Computing the right side of this formula using the hypergeometric function calculation method described in section II.7.1 works well for values $z > 1300$.

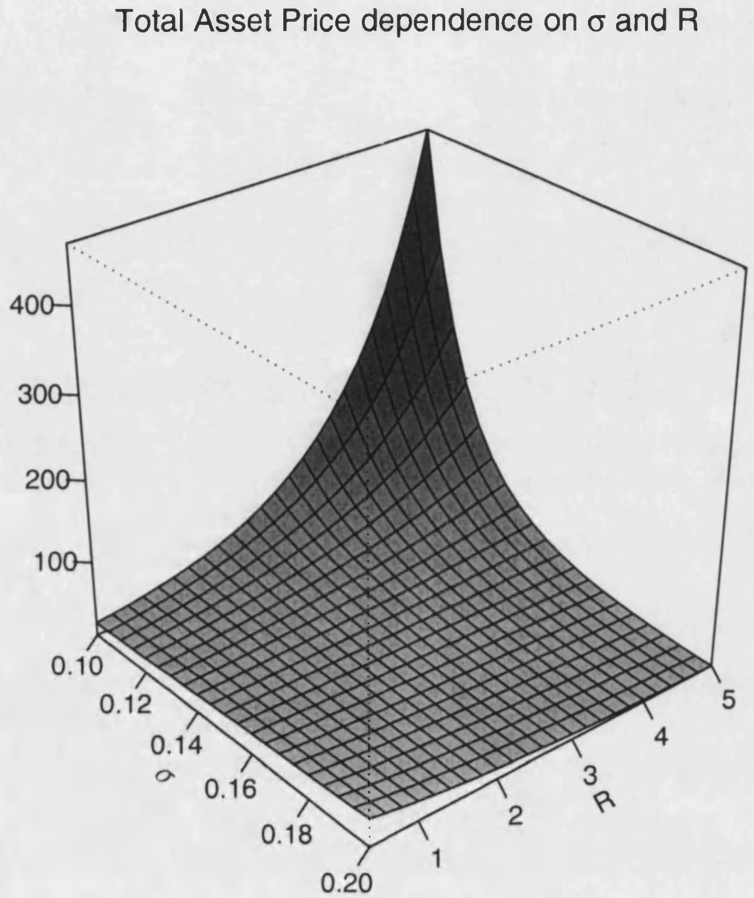


Figure II.3: The surface represents the total asset value as a function of risk aversion, R, and of the volatility, σ, of the aggregate output Δ. Risk aversion ranges from 0.4 to 5, while σ varies between 0.1 and 0.2. Remaining model parameters are as in Table II.2, and the level of Δ is fixed at 2.5.

For purposes of illustration, we fixed the strike price $K = f(A/\beta)$ to correspond to the market level when Δ is at the mean of its stationary law, and computed call prices, put prices, and the implied volatility surface. The latter is shown in Figure II.4. For comparison, we also plot volatility of yields of bonds of same lifetimes as the options (Figure II.5), as given by the expression (II.6.4).

The put-call parity relation for our model is, in obvious notation,

$$\begin{aligned}\mathrm{Call}(\Sigma_0, T, K) - \mathrm{Put}(\Sigma_0, T, K) &= \mathbb{E}^{\Delta_0}[\zeta_T \Sigma_T/\zeta_0] - K\mathbb{E}^{\Delta_0}[\zeta_T/\zeta_0] \\ &\equiv \tilde{\Sigma}_0 - K\exp(-\tilde{r}T).\end{aligned}$$

In the Black-Scholes model, the second line here would be the difference in price between a call and a put option written when the underlying is at $\tilde{\Sigma}_0$, with expiry time T and constant rate of interest $\tilde{r}$. In computing implied volatilities for our model, we therefore solved for the volatility parameter in the Black-Scholes pricing formula with $\tilde{\Sigma}_0$ as starting value for the asset, with $\tilde{r}$ being the yield of a bond expiring with the option, and with the dividend rate being zero.[15]

II.8. Conclusions

We have taken a simple and quite explicit model for a multi-asset single-agent economy in which the prices of bonds and shares can be computed in closed form, and simple recipes can be provided for pricing effectively any European option. The one-factor interest-rate model implied by the model is of an apparently novel form, and we have fitted the model to yield curve data.

The quality of fit that we obtain is reasonable, and the stability of parameter estimates is good. Our principal aim in calibrating the model was to obtain typical parameter values to work with when investigating the implications of the model assumptions on asset price dynamics. However, in principle, the time series of parameters obtained in fitting the model might also give information about the dynamics of risk aversion and the market price of risk over the period fitted. Indeed, the values for R and ρ that we obtain are sensible.

We have computed implied volatilities for European call options, and find that these typically exhibit a skew, not unlike actual data. The basic model has

[15]... dividends having already been taken into account implicitly in computing $\tilde{\Sigma}_0$.

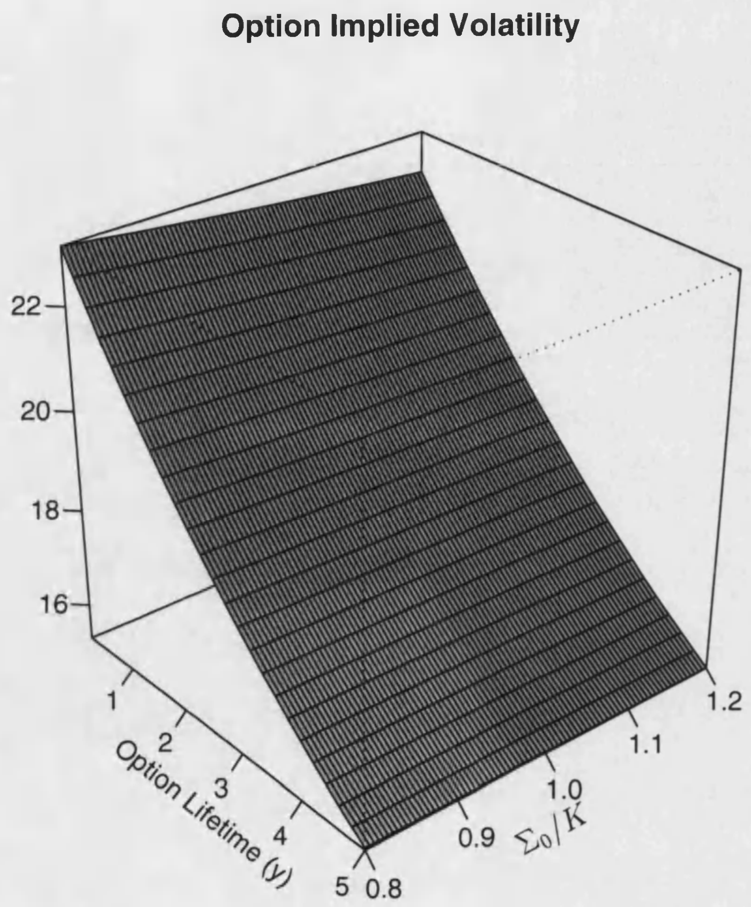


Figure II.4: Implied Volatility (%) surface for European call options on the total asset value Σ. Model Parameters are as in Table II.2. Strike $K = 35$; this value is close to the total asset price when Δ is at the mean of its stationary law. Option lifetimes range from 0.2 to 5 years, and the moneyness of the option varies from 80% to 120%.

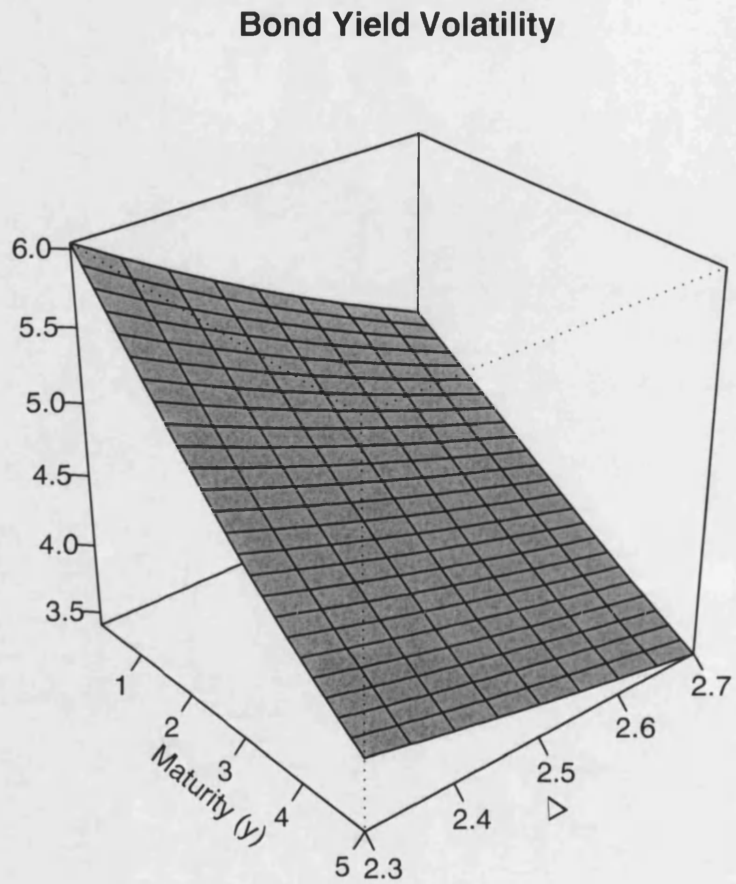


Figure II.5: Volatility (%) of bond yields as a function of Δ and maturity. Δ varies within one standard deviation away from the mean of its stationary law, and maturity times range from 0.2 to 5 years. Model parameters are as in Table II.2.

features in common with the CEV stock model with exponent 3/4, and such skewed implied volatility curves typically arise for CEV models.

The model assumptions are very restrictive; independence of the productive assets, and common volatility and mean-reversion parameters are quite severe. Nonetheless, under these assumptions we get a long way: we have built a consistent complete market model for multiple shares and the riskless rate.

The assumed CRRA utility of the agent can be relaxed a little. Indeed, we could as easily compute prices using as a state-price density

$$\zeta_t = a_1 e^{-\rho_1 t} \Delta_t^{-R_1} + a_2 e^{-\rho_2 t} \Delta_t^{-R_2}$$

for positive constants a_1, a_2, R_1, R_2, ρ_1, ρ_2. The point of this is not to aggregate across two heterogeneous agents, and indeed such aggregation with CRRA utilities of different coefficients of risk aversion results in no closed-form representative agent utility [16]. Rather, one would suppose that the aggregation procedure has led to a marginal utility that is expressible as the sum of two terms displayed above. Such an extension would allow for different coefficients of risk aversion for large and small consumption levels, lending some more flexibility to the model.

The non-negativity constraint on consumption is never binding in our model, due to the Inada condition satisfied at the origin by our CRRA-form utility. It is interesting to ask how the equilibrium solution we have computed would be affected if the utility and the market clearing condition were to be changed so as to allow the possibility of zero consumption. For instance, one could allow the representative agent to enter and exit the market (that is, consume all or none of the aggregate output) as he deems optimal, and the utility U could be extended to $[0, \infty)$ by setting $U(0) := u_0 > \lim_{t \downarrow 0} U(t)$, where $u_0 < U(\infty)$. Given such u_0, one could instead replace U by its concave majorant

$$\tilde{U}(z) := \begin{cases} u_0 + mz, & z \in [0, z^*], \\ U(z) & \text{otherwise,} \end{cases}$$

with z^* and m being chosen to make $\tilde{U}$ and its first derivative continous at z^*. Karatzas et al. (1986) solve in quite general terms the consumption / investment decision problem of a single agent faced with log-Brownian asset prices, taking into account the possibility of zero consumption. Whether reasonably explicit

[16]See, for example, Hara & Kuzmics (2002) for more detail on this point.

solutions can be obtained in an equilibrium setting is one avenue for futher work that could be explored in our model.

Chapter III

Equilibrium Models for Dependent Defaults

III.1. Introduction

It is a well-documented and unsurprising fact that the propensities to default of different firms do not generally evolve independently of one another. Moreover, in extreme situations the downfall of one firm may, through a contagious effect, quickly bring about a cascade of other defaults. The ways in which extant models incorporate such phenomena share the unsatisfying feature that the form of dependence between defaults is built in as part of the model rather than being derived endogenously from more elementary principles.

In what follows, we propose to view defaults of firms as the outcome in equilibrium of a rational decision on the part of shareholders in the firm. Share ownership entitles agents with given preferences to a fraction of the stochastic output from the firm. When outputs are large and positive, agents reap dividends; they may even be willing to inject capital (receive negative dividend) to keep a firm running in anticipation of larger dividends in future. However, for any particular set of firms, the outflow of capital required can become so large that it would be optimal for shareholders to exercise limited liability rights and downsize or shut down the firms. Such an equilibrium has the attractive features that (i) default is an endogenous event derived only from agents' preferences and dynamics of the output process, (ii) the effect of default of one firm on the propensity to do so of another arises endogenously also, and (iii) there is no reason a priori to exclude contagious effects.

The correct way to model multiple defaults is still a point of contention in the credit risk literature, and most attempts seem to fall within the class of so-called reduced-form models. The main feature of these is that default happens at the first jump time of a Poisson Process (with generally stochastic intensity). Jarrow and Turnbull (1995) specify exogenous term structures for both defaultable and default-free debt and assume that default occurs after an exponential length of time. By assuming independence between the default event and the default-free term structure, they obtain arbitrage prices for bonds and bond options. This approach is extended in Jarrow, Lando and Turnbull (1997) by modelling transition between rating classes as a Markov Chain, default being an absorbing state of the chain. Lando (1998) does away with the independence assumption and allows the stochastic intensity of the default-triggering jump process to be a function of state variables of the economy, similar in spirit to the approach of Duffie and Singleton (1999)[1]. The foregoing models only consider one firm, but the same idea can be applied in a multi-firm model, introducing dependence between defaults by allowing the intensity processes for different firms to depend on the same state variable(s). If nothing else is done, defaults remain conditionally independent, and to get around this Jarrow and Yu (2001) introduced the notion of counterparty risk, whereby the intensity process of a firm may jump at time of default of other firms. Copula-based models such as Li (2000) and Schönbucher and Schubert (2001) attempt to link the distribution of default times, conditional on the economy state variable, through some specified multivariate distribution. In an interesting approach, Khadem and Perraudin (2001) study an equilibrium with two firms where each firm has incentive to not be the first firm to default, thereby gaining a monopoly advantage. The authors derive endogenously the intensity rates that arise in equilibrium when both firms are either identical or else differ and have incomplete information about each other's characteristics.

In the class of structural models stemming from Merton (1974), one typically postulates dynamics for the value process of firms. Default is then triggered when this process reaches some critical barrier that needs to be determined. Black and Cox (1976) study a log-Brownian asset and deterministic exponential barriers.

[1]These papers also note that pricing expressions for defaultable claims are of the same form as for default-free ones, albeit with different discount rate. If term structure is chosen to be affine, explicit pricing expressions become available, allowing calibration to observed term structure, for example.

Leland (1994) and Leland and Toft (1996) take similar dynamics with a constant barrier. In Leland (1994) firm debt is perpetual and attracts a constant coupon rate, while Leland and Toft (1996) assume a specific debt maturity profile; both situations are consistent with a constant default barrier, which is chosen endogenously to maximize the value of the firm's equity. Rogers and Hilberink (2002) extended this approach to allow (one-sided) jumps in the value process. With multiple assets in structural models, default dependence arises from correlation between value processes of different firms *as well as* from the form chosen for the critical boundary. Zhou (2001) studies a market with two firms having correlated log-Brownian dynamics. Default is characterized by the event that a pair of correlated Brownian Motions exits the upper-right orthant in the plane, and the only motive for this choice seems to be that the correlation between the default indicator processes becomes reasonably explicit. Giesecke (2002, 2003), who extends the one-firm model of Duffie and Lando (2001), starts from the notion that firm value and default-triggering barriers are a priori unknown to bond investors. These barriers are further linked through a given copula function. As information about defaults arrives, bondholders update their beliefs about default barriers pertaining to non-defaulted firms, causing prices of defaultable bonds to change. In contrast to Zhou's model, the simultaneous default of more than one asset can occur in this information-based model.

Our goal differs from those of the above-mentioned studies; we do not aim to price default-sensitive instruments or fit observed term structure of credit spreads. Rather, we abstract from capital structure, debt and debt provisions and view firms purely as an entitlement to an output or liability flow[2]. We then investigate an equilibrium with limited liability where shareholders may rationally choose to reduce the scale of operation of firms or to halt it altogether. In particular, we ask whether such an equilibrium can sustain contagion.

To fix some ideas about the kind of effects we have in mind, consider the following static one-period example, where a sensible equilibrium may exist only after elimination of assets. Suppose we have $N+1$ assets. The zeroth asset is worth 1 at times 0 and 1. Amounts $A \in \mathbb{R}^N$ of shares in the remaining assets are available to start with, worth $S_0 \in \mathbb{R}^N$ at time 0 and a normally-distributed vector $S_1 \sim N(\mu, V)$ at time 1. Here $\mu \in \mathbb{R}^N$ and V is a symmetric positive definite $N \times N$ matrix.

[2]It is not too difficult, however, to see how such considerations as firm debt and capital structure could be embedded in the framework presented here.

Consider J agents who are initially each in possession of non-negative amounts of shares. Assume that agent j, having a utility function $U_j : x \mapsto -\exp(-\gamma_j x)$ with $\gamma_j > 0$ chooses a portfolio θ_j of holdings in the assets so as to attain $\max \mathbb{E}\big[U_j(\theta_j \cdot S_1)\big]$. The optimal portfolio for j is easily found to be

$$\theta_j^* = \gamma_j^{-1} V^{-1}(\mu - S_0),$$

where we note that the initial asset distribution among agents does not matter.

The price vector S_0 is here taken as fixed, but if agents' demands are to clear the markets for the assets, whereby $A = \sum_j \theta_j^*$, then we must have

$$S_0 = \mu - \Gamma V A, \tag{III.1.1}$$

with $\Gamma^{-1} \equiv \sum_j \gamma_j^{-1}$. Depending on the distribution of S_1, some entries in the price vector S_0 may be negative, which is problematic given the price interpretation of S_0. In order to obtain an equilibrium with non-negative prices in this setup, some assets, or at least a proportion of shares of assets, need to be eliminated.

If V is diagonal, of course, decreasing asset supply of each negatively-priced asset causes prices of those assets to increase, and this can be done until either supply or price reaches zero. If, say, assets j and $k \neq j$ are negatively correlated, then reducing A_j will will have opposite effects on $S_{0,j}$ and $S_{0,k}$. Which assets to eliminate, and in which order, to arrive at a set of assets resulting in non-negative equilibrium prices is not clear in general. However, we can obtain a valid equilibrium concept for this setup by allowing prices to be Lagrangian multipliers ('shadow prices') of an appropriate quadratic program. To this end, consider

$$\min \left\{ z \cdot [-\mu + \frac{1}{2} \Gamma V z] \right\} \quad \text{subject to } A \geq z \geq 0. \tag{III.1.2}$$

Because of the assumptions on V, a unique solution z^* exists (see, for example, Cottle, Pang and Stone 1992) which satisfies

$$\begin{aligned} u &:= (-\mu + \Gamma V z^*) + \lambda \geq 0, & z^* &\geq 0; \quad u \cdot z^* = 0, \\ v &:= (A - z) \geq 0, & \lambda &\geq 0; \quad v \cdot \lambda = 0, \end{aligned} \tag{III.1.3}$$

for some $\lambda \in \mathbb{R}^N$. We then have the implications

$$\begin{cases} A_k > z_k^* > 0 & \Longrightarrow \lambda_k = (\mu - \Gamma V z^*)_k = 0; \\ z_k^* = A_k & \Longrightarrow \lambda_k = (\mu - \Gamma V z^*)_k \geq 0; \\ z_k^* = 0 & \Longrightarrow \lambda_k = 0, \quad (\mu - \Gamma V z^*)_k \leq 0. \end{cases}$$

If we interpret z^* as the amount of shares retained, then $\lambda_k = (\mu - \Gamma V z^*)_k \geq 0$ is a valid non-negative equilibrium price for asset k consistent with the form (III.1.1). If $z_k^* = 0$, then $(\mu - \Gamma V z^*)_k$ may be negative, but this is irrelevant as the asset exists no more in this case.

The foregoing example is of course highly stylized and hardly realistic, but the point remains that a mechanism appears whereby *in equilibrium*, some assets are eliminated and those remaining lend value to each other. We shall now endeavour to reproduce a similar mechanism in a dynamic setting.

III.2. General Model

We take as our fundamental object a strong Markov process $X \equiv \{(X_t),\ t \in \mathcal{T}\}$, defined on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathcal{T}}, \mathbb{P})$[3] and taking values in $\mathbb{R}^N$, with the index set being[4] $\mathcal{T} = \mathbb{R}^+$ or $\mathcal{T} = \mathbb{Z}^+$. For each k, $1 \leq k \leq N$, we think of there being an asset (firm) k, an entitlement to the output / liability flow represented by the k'th component of X. We shall write $\mathcal{N}$ for the set of N assets and use similar calligraphic notation to denote subsets thereof.

There are J agents owning the firms, the j'th[5] agent having exponential utility function

$$U_j : x \mapsto -\exp(-\gamma_j x)$$

of CARA $\gamma_j > 0$. Starting with holdings $\phi_j(0-) \in \mathbb{R}^N$ in shares of the firms, agent j maintains a portfolio process $\phi_j \equiv \{\phi_j(t), t \in \mathcal{T}\}$ entitling him at time $t \in \mathcal{T}$ to a cashflow rate $\phi_j(t) \cdot X_t$. By possibly borrowing or lending against future income to have a net cashflow $c_j \equiv \{c_j(t), t \in \mathcal{T}\}$, he enjoys accumulated

[3]... satisfying the usual conditions. The augmented filtration $(\mathcal{F}_t)_{t\in\mathcal{T}}$ is assumed to be that generated by X.

[4]By $\mathbb{R}^+$ we shall mean $[0, \infty)$ and by $\mathbb{Z}^+$ the set $\{0, 1, 2, \ldots\}$.

[5]We employ the convention of using the subscript j to represent a generic agent, and subscript k to represent a generic firm.

utility

$$J_j(x, \phi_j, c_j; \theta_j) := \mathbb{E}\Big[\int_0^\infty e^{-\delta_j t} U_j(c_j(t))\, dt \Big| X_0 = x,\ \phi_j(0-) = \theta_j\Big] \qquad \text{(III.2.1)}$$

when $\mathcal{T} = \mathbb{R}^+$ and

$$J_j(x, \phi_j, c_j; \theta_j) := \mathbb{E}\Big[\sum_{n=0}^{\infty} \beta_j^n U_j(c_j(n)) \Big| X_0 = x,\ \phi_j(0-) = \theta_j\Big] \qquad \text{(III.2.2)}$$

in the case $\mathcal{T} = \mathbb{Z}^+$. Here, $\delta_j > 0$ and β_j, $0 < \beta_j < 1$, are discount factors, and the formal notation '$0-$' is to take account of the fact that the holdings $\phi_j(0)$ may differ from the initial allocation θ_j. *We assume $\theta_j > 0$ for all j.*

We insist on the market-clearing condition that the total firm output be entirely accounted for among agents, that is, the processes c_j and ϕ_j, $1 \leq j \leq J$, satisfy[6]

$$\sum_j c_j(t) = \sum_j \phi_j(t) \cdot X_t. \qquad \text{(III.2.3)}$$

The j'th agent's objective is to choose a budget-feasible process c_j and a portfolio process ϕ_j so as to attain the value function

$$v_j(x, \theta_j) = \max_{c_j,\ \phi_j \in \varphi_j} J_j(x, \phi_j, c_j; \theta_j), \qquad \text{(III.2.4)}$$

where the class φ_j consists of portfolios for which j's wealth remains bounded from below. We assume moreover that the finiteness conditions

$$\mathbb{E}^x\big[\int_0^\infty e^{-\delta_j t} U_j(X_t) dt\big] > -\infty, \qquad \mathbb{E}^x\big[\sum_{n=0}^{\infty} \beta_j^n U_j(X_n)\big] > -\infty, \quad \forall x \in \mathbb{R}^N,\ \forall j, \qquad \text{(III.2.5)}$$

hold, implying finiteness of v_j for each $x \in \mathbb{R}^N$; algebraic conditions that guarantee (III.2.5) will be given.

Agents have *limited liability* in the sense that in equilibrium the aggregate share amount process $\Phi \equiv \{\Phi(t),\ t \in \mathcal{T}\}$ defined by $\Phi(t) \equiv \sum_j \phi_j(t)$ may not be

[6]Because the processes c_j will not be restricted to be non-negative, one cannot honestly talk of c_j as being a 'consumption' process for agent j. We shall see, however, that under the optimal strategy for j, the processes c_j remain bounded below. We can therefore suppose that agent j is endowed further with an exogenous labour income stream that stays constant over the agent's infinite lifetime and that makes the agent's total consumption non-negative. The effect of this is to multiply the agent's exponential utility by a constant factor, which, as we shall see, leaves unchanged the optimal strategy employed by the agent.

identically equal to $\Phi(0-) \equiv \sum_j \phi_j(0-)$ and may be decreasing in general. We say that default of firm k occurs at time

$$\inf\{t : \Phi_k(t) = 0\}.$$

The ways in which defaults happen in equilibrium is what we want study.

In fact, we shall assume in what follows that there is one (representative) agent in the market, having exponential utility $U_1 \equiv U$ of CARA $\gamma_1 \equiv \Gamma > 0$. This trivializes the market clearing condition (III.2.3), but it has the simplifying effect of making our object of interest, the process $\phi_1 \equiv \Phi \in \varphi_1 \equiv \varphi$, the only choice variable[7]. With this simplification, we have the obvious lower bounds

$$v(x,\theta) \geq U(0)/\delta; \quad v(x,\theta) \geq U(0)/(1-\beta), \tag{III.2.6}$$

where we have dropped the now irrelevant subscript j identifying agents. We expect, and we shall prove for our concrete examples, that the value function v is increasing in its first argument.

Because of (III.2.3), with only one agent present, the portfolio process Φ is simply the aggregate share amount in the economy, which can be altered (i.e. *decreased*) because of limited liability. The class $\varphi_1 \equiv \varphi$ appearing in (III.2.4) can therefore be interpreted as an admissible class of non-negative controls adapted to the filtration $(\mathcal{F}_t)$ of the aggregate output X. For the most part, we shall take $\Phi \in \varphi^s$, where

$$\begin{aligned}\varphi^s(\theta) := \{\phi : \phi_k \geq 0,\ (\mathcal{F}_t)\text{-adapted, non-increasing,}\\ \phi_k(t) = 0 \text{ or } \phi_k(t) = \theta_k\ \forall t \in \mathcal{T},\ 1 \leq k \leq N\}.\end{aligned} \tag{III.2.7}$$

The control exerted by the single agent in this case is evidently to either maintain the initial amount θ of shares available or to relinquish these shares entirely. Under restrictive conditions on the process X in a one-firm market, we will have occasion to solve (III.2.4) with $\Phi \in \varphi^c$, where

$$\begin{aligned}\varphi^c(\theta) := \{\phi : \phi_k \geq 0,\ (\mathcal{F}_t)\text{-adapted, non-increasing, continuous,}\\ \phi_k(t) \in [0,\theta_k]\ \forall t \in \mathcal{T},\ 1 \leq k \leq N\},\end{aligned} \tag{III.2.8}$$

[7]See Appendix A.4 for a genuinely multi-agent equilibrium computation in a one-firm market.

the allowed control now being a *gradual* downsizing of firms.

In summary, the strategy employed by the agent will be to choose, at any $t \in \mathcal{T}$, to decrease or not the current share amount $\Phi(t)$; the time of default for firm k is the time when Φ_k reaches zero. Because $\Phi(t)$ is assumed to be $\mathcal{F}_t$-measurable for each $t \in \mathcal{T}$ and U is increasing, we expect default to be suboptimal in a *continuation set*[8] $\mathbb{C}^* := \mathbb{R}^N \setminus \mathbb{S}^*$ where all components of X are large. Default is triggered when X exits this region, and the problem is to characterize the *exercise boundary* $\overline{\mathbb{C}^*} \cap \overline{\mathbb{S}^*}$ that triggers defaults and to understand how defaults happen once this exercise boundary is breached. In what follows, we look for answers to these questions in the context of particular examples for the process X.

III.3. The discrete-time problem with IID output

We start by studying the value function (III.2.4) corresponding to the discrete-time objective (III.2.2). The key simplifying assumption we make is that $X \equiv \{X_n,\ n \in \mathbb{Z}^+\}$ is a sequence of IID random variables. In fact, for computations we shall assume that *the X_n are multivariate normal random variables*, and with no loss of generality we assume also that either one or no share of any firm is available at any time. That is, the maximization in (III.2.4) is over the class $\varphi^s(\Phi(0-))$ with $\Phi(0-) = I_{\mathcal{A}}$ being the indicator function of a subset $\mathcal{A} \subseteq \mathcal{N}$.

Carrying out the maximization, we obtain the dynamic programming equation

$$\begin{aligned} v(x, I_{\mathcal{A}}) &= \max_{\tilde{\mathcal{A}} \subseteq \mathcal{A}} \left\{ U(x \cdot I_{\tilde{\mathcal{A}}}) + \beta \mathbb{E} v(X_1, I_{\tilde{\mathcal{A}}}) \right\} \\ &= \max \left\{ U(x \cdot I_{\mathcal{A}}) + \beta K_{\mathcal{A}},\ \max_{\tilde{\mathcal{A}} \subset \mathcal{A}} v(x, I_{\tilde{\mathcal{A}}}) \right\} \end{aligned} \tag{III.3.1}$$

Here, we have set $K_{\mathcal{A}} := \mathbb{E} v(X_1, I_{\mathcal{A}})$, and the second equality is a consequence of the IID assumption. We see, then, that the solution of the N-firm problem entails knowing only the 2^N constants $\{K_{\mathcal{A}},\ \mathcal{A} \subseteq \mathcal{N}\}$.

As is evident from (III.3.1), the agent owning the single share of each of the firms in some subset $\mathcal{A}$ decides at each time $t \in \mathcal{T}$ whether or not to default one or more firms. Thus, we obtain a partition $\mathbb{C}^*_{\mathcal{A}} \cup \mathbb{S}^*_{\mathcal{A}} = \mathbb{R}^N$ of the range of X_1 such that $x \in \mathbb{C}^*_{\mathcal{A}}$ if, and only if, the maximand in (III.3.1) is the first term in braces. We refer to $\mathbb{C}^*_{\mathcal{A}}$ and $\mathbb{S}^*_{\mathcal{A}}$ as the *continuation set* and *stopping set*,

[8] We assume from now on that $\mathbb{S}^*$ and $\mathbb{C}^*$ are both connected.

respectively, pertaining to $\mathcal{A}$. Because U is continuous, the value function (III.3.1) is continuous in $x \in \mathbb{C}^*_{\mathcal{A}}$; therefore, on the *exercise boundary* $\overline{\mathbb{C}^*_{\mathcal{A}}} \cap \overline{\mathbb{S}^*_{\mathcal{A}}}$ that triggers default, the agent is just indifferent between choosing to default or not.

III.3.1. The one-firm problem

If $\mathcal{N} = \{1\}$, then the form of (III.3.1) is immediate, and we get that

$$\begin{aligned} v(x,1) &= \max\Big\{U(x) + \beta\mathbb{E}v(X_1,1), U(0)/(1-\beta)\Big\} \\ &= \begin{cases} U(x) + \beta K, & \text{if } x \geq b^*; \\ U(0)/(1-\beta), & \text{if } x < b^*, \end{cases} \end{aligned} \tag{III.3.2}$$

where $b^* \in \mathbb{R}$, the exercise boundary for this problem, is to be determined, the second equality follows because U is increasing and continuous, and $K \equiv K_{\{1\}}$.

Now whatever b^* is we shall have

$$K = \mathbb{E}[U(X_1) + \beta K;\ X_1 \geq b^*] + \mathbb{P}(X_1 < b^*)U(0)/(1-\beta)$$

so the constant K satisfies

$$K[1 - \beta\bar{F}(b^*)] = \mathbb{E}[U(X_1); X_1 \geq b^*] + U(0)F(b^*)/(1-\beta), \tag{III.3.3}$$

where $F \equiv 1 - \bar{F}$ is the distribution function of X_1. In order to determine b^*, we solve

$$v(b^*,1) = U(b^*) + \beta K = U(0)/(1-\beta),$$

and incorporating (III.3.3) allows us to characterize b^* as the solution to

$$U(b)[1 - \beta\bar{F}(b)] + \beta\mathbb{E}[U(X_1); X_1 \geq b] = U(0). \tag{III.3.4}$$

We now invoke the Gaussian distributional assumption $X_1 \sim N(\mu, \sigma^2)$ and the exponential form for U to get the particular form

$$e^{-\Gamma b^*}\big(1 - \beta\bar{\Phi}((b^* - \mu)/\sigma)\big) + \beta e^{\Gamma(\Gamma\sigma^2/2-\mu)}\bar{\Phi}\left(\frac{b^* - (\mu - \Gamma\sigma^2)}{\sigma}\right) = 1, \tag{III.3.5}$$

where $\Phi \equiv 1 - \bar{\Phi}$ is the standard normal distribution function.

III.3.2. Numerical solution for $N > 1$

As we pointed out, all we need to know for $N > 2$ is a finite number of constants $\{K_{\mathcal{A}},\ \mathcal{A} \subseteq \mathcal{N}\}$. To proceed inductively, take $\mathcal{A} \subseteq \mathcal{N}$. If $\mathcal{A}$ is a singleton, then we are in the one-firm situation discussed above. Otherwise, suppose $K_{\tilde{\mathcal{A}}}$, and hence $v(x, I_{\tilde{\mathcal{A}}})$, is known for each $\tilde{\mathcal{A}} \subset \mathcal{A}$.

Then, the constant $K_{\mathcal{A}}$ can be computed numerically using a value-improvement scheme that yields a monotonic increasing sequence $(K^m_{\mathcal{A}}) \to K_{\mathcal{A}}$, as follows. Let

$$K^0_{\mathcal{A}} = \mathbb{E}\Big[\max_{\tilde{\mathcal{A}} \subset \mathcal{A}} v(X_1, I_{\tilde{\mathcal{A}}})\Big],$$

and given $K^{m-1}_{\mathcal{A}}$ for $m \geq 1$ define

$$v_m(x, I_{\mathcal{A}}) = \max\Big\{U(x \cdot I_{\mathcal{A}}) + \beta K^{m-1}_{\mathcal{A}}, \max_{\tilde{\mathcal{A}} \subset \mathcal{A}} v(x, I_{\tilde{\mathcal{A}}}).\Big\}$$

Now set

$$K^m_{\mathcal{A}} := \mathbb{E} v_m(X_1, I_{\mathcal{A}}),$$

which satisfies $K_{\mathcal{A}} \geq K^m_{\mathcal{A}} \geq K^{m-1}_{\mathcal{A}}$. This latter inequality is tight if, and only if, $K^{m-1}_{\mathcal{A}} = K_{\mathcal{A}}$. The bounded increasing sequence $(K^m_{\mathcal{A}})$ thus constructed therefore converges to the required value $K_{\mathcal{A}}$, and in practice we continue iterating the procedure until $\|K^m_{\mathcal{A}} - K^{m+1}_{\mathcal{A}}\|$ is tolerably small.

There are at least two different ways in which the expectation involved in each step can be computed, one by Monte-Carlo simulation and the other by performing numerical integration on a fine grid.

III.3.3. Occurrence of default

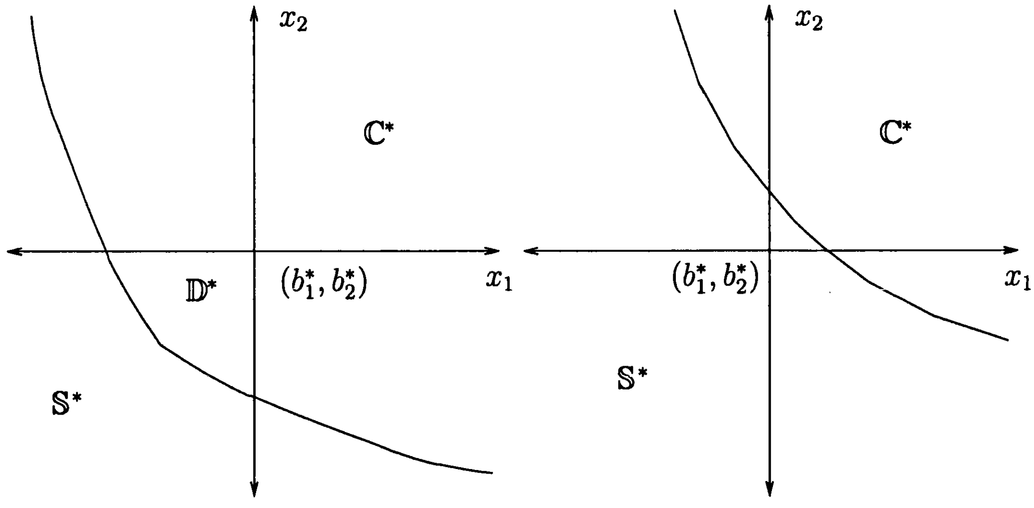
The way defaults occur in this model is prescribed by the exercise boundary, and one particular form this could take has interesting implications.

Set

$$\Psi(x, I_{\mathcal{A}}) := \max_{\tilde{\mathcal{A}} \subset \mathcal{A}} v(x, I_{\tilde{\mathcal{A}}}),$$

the maximum attainable if the agent had to default at least one firm from the set $\mathcal{A}$. Now consider the situation where for some $x \in \mathbb{C}^*_{\mathcal{A}}$ we have

$$\Psi(x, I_{\mathcal{A}}) = v(x, I_{\mathcal{C}}), \quad \mathcal{C} \subset \mathcal{A}, \quad |\mathcal{C}| < |\mathcal{A}| - 1.$$



(a) Contagion region non-empty. (b) Contagion region empty.

Figure III.1: Two possibilities for location of exercise boundary. Origin of co-ordinates is the point (b_1^*, b_2^*) where b_k, $k = 1, 2$ is the exercise boundary for the one-firm model for firm k.

This is a symbolic way of saying that at x, it is suboptimal to default any firm, but if the agent *were* forced to default, then *at least two* firms would be defaulted at once. For each such x, then, firms would have value only by virtue of their being owned together. For this reason, we shall call the set

$$\mathbb{D}_{\mathcal{A}}^* := \{x \ : \ v(x, I_{\mathcal{A}}) > \Psi(x, I_{\mathcal{A}}), \ \Psi(x, I_{\mathcal{A}}) = v(x, I_{\mathcal{C}}), |\mathcal{C}| < |\mathcal{A}| - 1\}, \quad \text{(III.3.6)}$$

the *contagion region*, and to establish whether this region can be non-empty, we solved numerically for several instances of the model with $N = 2$ firms.

To spell out what we should be looking for in this case, note that with $\mathcal{N} = \{1, 2\}$, we have

$$\mathbb{D}^* \neq \emptyset \iff (b_1^*, b_2^*) \in \mathbb{C}^*, \quad \text{(III.3.7)}$$

where b_k is the exercise boundary derived in a one-firm model for firm k, $k = 1, 2$, and where we have omitted the subscript $\mathcal{N}$. Figure III.1 shows (III.3.7) pictorially. Results given in Section III.6.1 present several model parameters for which $\mathbb{D}^*$ is indeed non-empty, and this motivated us to re-work the model allowing X to be a continuous process. If we were to think of X as drifting continuously from a non-empty contagion region towards the exercise boundary that triggers default, then we would see several firms being defaulted simultaneously, so a non-empty contagion region with continuous X would imply that contagious effects occur in the equilibrium we have computed.

III.4. The continuous-time problem with Lévy output: the one-firm case

With only one firm present, our model represents a one-dimensional optimal stopping problem (characterized, in fact, by a stationary exercise strategy). Problems of this kind are the subject of a large body of literature in finance, including, of course, the well-known American option pricing problem. Hilberink and Rogers (2002) extend the structural-type credit-risk model of Leland (1994) and Leland and Toft (1996) by modelling the value process of a firm as a spectrally negative Lévy process. Given a clever choice for the debt profile of the firm, the stopping problem faced by shareholders is to choose the level below which the firm is to be declared bankrupt; the choice of this critical boundary is made to maximize the value of the firm's equity. Miao and Wang (2004) consider an entrepreneur who needs to decide when to undertake an investment project. There is a cost associated with embarking on the project, which then generates a cashflow modelled as Brownian motion with drift. The entrepreneur's decision is timed so as to maximize infinite-lifetime expected utility of consumption, while wealth may be invested in a risky asset as well as a riskless bank account. By using a utility of CARA form, the authors obtain the form of the exercise boundary for the problem, and this is found to depend on the cashflow ensuing from the investment but not on the entrepreneur's wealth level. The availability of closed-form solutions allows several interesting comparative statics to be performed, and the effects on the solution of changes in risk aversion are examined. Kadam et al. (2004) study the problem faced by the holder of a perpetual American option when the option's underlying - whose dynamics are log-Brownian - cannot be traded. They choose their exercise time to maximize the utility of the option's payoff, with the utility function chosen to be of CRRA form. The pricing of convertible defaultable bonds is also at heart an optimal stopping problem; see, for example, Bermudez and Webber (2003) and references therein. Also in Insurance Mathematics, the search for optimal strategies that control an insurer's cashflow to maximize dividend value and / or minimize ruin probability often leads to first-passage problems similar in nature to those arising in optimal stopping. See, for example, Hojgaard and Taksar (1999), and Asmussen et al. (2000) for reasonably explicit solutions to the problem of how a risk-netural insurer should behave in effecting reinsurance while maximizing dividend value.

Consider now our particular continuous-time objective (III.2.1) when $N = 1$ and $\mathcal{T} = \mathbb{R}^+$. This is to be maximized to attain value

$$v(x,1) = \max_{\Phi \in \varphi^s(1)} \left[\mathbb{E}^x \int_0^\infty e^{-\delta t} U(\Phi_t X_t)\, dt \right]; \tag{III.4.1}$$

because U is increasing and Φ is restricted to lie in $\varphi^s(1)$, we have equivalently

$$v(x,1) = \max_{b \in \mathbb{R}} \mathbb{E}^x \left[\int_0^{\tau_b} e^{-\delta t} U(X_t)\, dt + e^{-\delta \tau_b} U(0)/\delta \right], \tag{III.4.2}$$

where for any $a \in \mathbb{R}$, $\tau_a := \{t : X_t < a\}$ is the first time X enters $(-\infty, a)$. The maximizing b^* in (III.4.2) is the exercise boundary for this problem, specifying that the optimal strategy is to take $\Phi^* \equiv 0$ if, and only if, $x < b^*$. The continuation and stopping sets are therefore $\mathbb{C}^* = [b^*, \infty)$ and $\mathbb{S}^* = (-\infty, b^*)$, respectively[9], with the interpretation that the share ownership is relinquished and the firm defaulted as soon as X enters $\mathbb{S}^*$. For this one-dimensional problem, we shall characterize b^* and obtain an expression for the value (III.4.2) in fairly explicit form. In the examples we treat, $v(\cdot, 1)$ will be continuous in (b^*, ∞), and *also at* b^*.

The canonical example here is to have $X = \{X_t,\ t \geq 0\}$ a Brownian Motion, but the analysis can be carried through for a one-sided Lévy Process and this is the more general setting we work in. If the exponential form of U is relaxed and X is a diffusion, then the exercise boundary b^* can still be computed; see Section III.4.7. If X is Brownian Motion, Brownian excursion techniques help us obtain the value function with Φ ranging over the class $\varphi^c(\cdot)$, which we do in Section III.4.5.

[9]Note that the stopping set $\mathbb{S}^*$ is chosen to be open. Which of $\mathbb{C}^*$ and $\mathbb{S}^*$ is open is inconsequential when X enters $(-\infty, b^*)$ immediately from b^*, but our convention is necessary when 0 is not regular for $(-\infty, 0)$.

III.4.1. Characterization of the exercise boundary

If the output process X is a Lévy process, the exercise boundary b^* attaining the value (III.4.2) can be characterized in terms of the Wiener-Hopf factors of X. The Lévy exponent for such X,

$$\psi(z) = \frac{1}{t}\log \mathbb{E}^0[\exp(zX_t)], \tag{III.4.3}$$

is then well-defined on some domain $\mathcal{D} \supseteq i\mathbb{R}$ and independent of $t \in \mathbb{R}^+$, and has the Lévy-Khinchine representation

$$\psi(z) = \frac{1}{2}\sigma^2 z^2 + \mu z + \int_{\mathbb{R}\setminus\{0\}} (e^{zx} - 1 - z(|x| \wedge 1))\ \nu(dx), \tag{III.4.4}$$

where $\nu(\cdot)$, the Lévy measure of X, is a measure on $\mathbb{R} \setminus \{0\}$ satisfying the integrability condition

$$\int_{\mathbb{R}\setminus\{0\}} (x^2 \wedge 1)\ \nu(dx) < \infty.$$

For more details, see Bertoin (1996). Associated with X we have its infimum process $\underline{X} \equiv \{\underline{X}_t := \inf_{s\leq t} X_s\}$ and the analogously-defined supremum process $\overline{X}$.

Note, by multiplying (III.4.2) by δ, that the number b^* attaining (III.4.2) maximizes

$$\mathbb{E}^x\left[\int_0^{\tau_b} \delta e^{-\delta t} U(X_t)dt + e^{-\delta\tau_b}U(0)\right],$$

which can be written in terms of an exponential random variable T of rate δ, as

$$\mathbb{E}^0\left[U(x + X_T) - U(0)\ ;\ T < \tau_{b-x}\right] + U(0). \tag{III.4.5}$$

Invoking the exponential form of the function U, we get the above to look like

$$\begin{aligned} &\mathbb{E}^0\left[-\exp\left(\Gamma(x + X_T)\right) + 1\ ;\ T < \tau_{b-x}\right] - 1 \\ =&\mathbb{E}^0\left[-e^{-\Gamma x}e^{-\Gamma\overline{X}_T}e^{-\Gamma\underline{X}_T} + 1;\ \underline{X}_T \geq b - x\right] - 1, \end{aligned} \tag{III.4.6}$$

because $\overline{X}_T$ is identical in law to $X_T - \underline{X}_T$ and *independent* of $\underline{X}_T$. The identity in law is obvious, while independence is given us by the Wiener-Hopf factorization[10]

[10]See Rogers & Williams (2000, I.29), or Bertoin (1996).

of X. This says also that

$$\begin{aligned} \mathbb{E}^0[e^{zX_T}] &= \frac{\delta}{\delta - \psi(z)} \\ &= \mathbb{E}^0[e^{z\underline{X}_T}]\mathbb{E}^0[e^{z\overline{X}_T}] \\ &=: \psi^-(z)\psi^+(z), \end{aligned} \tag{III.4.7}$$

where ψ^+ (resp. ψ^-) is bounded and analytic in the left (resp. right) half of the complex plane; see Sato (1999, Ch. 9).

Invoking the factorization above and maximizing the expression (III.4.6) with respect to b, we obtain b^* as the solution to

$$e^{-\Gamma b} = 1/\psi^+(-\Gamma). \tag{III.4.8}$$

This is all very well, *modulo the computation of the Wiener-Hopf factor* ψ^+ for which no closed form expression exists in general.

If, however, we assume[11] that the process X is spectrally negative, it is a straightforward probabilistic argument[12] (see Bertoin (1996)) that $\overline{X}_T$ *is an exponential random variable*, so that for some $\beta^* > 0$,

$$\psi^+(z) = \frac{\beta^*}{\beta^* - z}; \tag{III.4.9}$$

therefore, knowing b^* amounts to knowing the rate β^*.

Now, substituting the form of the Wiener-Hopf factor ψ^+ into (III.4.7), we see that whatever β^* is, the other Wiener-Hopf factor has to satisfy

$$\psi^-(z) = \frac{\beta^* - z}{\beta^*}\frac{\delta}{\delta - \psi(z)}; \tag{III.4.10}$$

because ψ^- is analytic at the point z if $\mathrm{Re}(z) > 0$, this determines $\beta^* \equiv \beta^*(\delta)$ as the solution to

$$\delta = \psi(\beta). \tag{III.4.11}$$

We can now deduce

[11]... an assumption that does not exclude the important examples of Brownian Motion with possibly non-zero drift, as well as compound Poisson Processes with negative jumps ...

[12]An exactly similar argument would apply if the process were spectrally positive, of course, to yield an exactly analogous conclusion.

Proposition III.4.1. *If X is spectrally negative and has Lévy exponent ψ as in (III.4.3), then the exercise boundary b^* maximizing (III.4.2) satisfies*

$$e^{-\Gamma b^*} = \frac{\Gamma + \beta^*}{\beta^*}, \tag{III.4.12}$$

where β^* solves $\psi(\beta) = \delta$.

The number b^* is non-positive, uniquely determined from (III.4.12), and invariant under affine transformations of the utility U. If $-X$ is not a subordinator, then $b^* < 0$.

Proof. The identity (III.4.12) follows immediately from (III.4.8). The invariance of b^* is evident from (III.4.5). ■

Solving (III.4.11) for β^* is generally a non-trivial task that can be accomplished only numerically. For the concrete examples we treat here, β^* appears as the positive root of an appropriate quadratic equation and is therefore obtainable in closed form.

REMARK III.4.2. The problem (III.4.2) can be posed in terms of an integro-differential equation involving the infinitesimal generator of X. This gives an equivalent characterization of b^* that will be useful in Section III.4.7 and in the multi-asset model of Section III.5.

Suppose $x > b^*$ [13] and define $M \equiv \{M_t, t \geq 0\}$ by

$$M_t = \int_0^t e^{-\delta s} U(X_s) ds + e^{-\delta t} v(X_t, 1). \tag{III.4.13}$$

For each $t < \tau(b^*)$ it is (by definition of b^*) optimal to have $\Phi_t^* = \Phi(0-) = 1$. The martingale optimality principle then implies that $(M(t \wedge \tau(b^*)))_{t\geq 0}$ is a $\mathbb{P}^x$-martingale. Applying Itô's lemma to M and remembering the appropriate lower bound (III.2.6), we see that $v(x) \equiv v(x, 1)$ satisfies complementarity problem:

$$\min_b \Big\{ (\delta - \mathcal{G}) v(x) - U(x), v(x) - \ U(0)/\delta \Big\} = 0, \tag{III.4.14}$$

which is the HJB equation for this one-firm problem, $\mathcal{G}$ being the infinitesimal generator of X. By definition, the minimand in (III.4.14) is the first term in braces if, and only if, $x > b^*$.

[13] Indeed, by virtue of Proposition III.4.1, any $x > 0$ satisfies this...

III.4.2. Computing the value function

Having obtained a general characterization for the exercise boundary, and hence for the maximizing process Φ^* in (III.4.1), we now strive to obtain some reasonably explicit representation for the value (III.4.2) itself. Except for Section III.4.5 we shall restrict the control process Φ to the class φ^s. In Section III.4.7 we relax the assumption on the form of U and impose instead continuity on the process X, resulting in an alternative characterization of the exercise boundary b^*.

Before we go further, we prove the intuitive fact that the value function $v(x) \equiv v(x,1)$ in (III.4.2) is increasing. For a Lévy process, this is a direct consequence of space-homogeneity. We use a coupling argument to prove the assertion also for the case when X is a one-dimensional diffusion.

Proposition III.4.3. *Assume U is strictly increasing. If the process X is either a one-dimensional diffusion or a Lévy Process on $\mathbb{R}$, then the value function (III.4.2) is strictly increasing in $\{x \geq b^*\}$.*

Proof. Suppose first that X is a Lévy Process on $\mathbb{R}$. If $\xi > 0$ and $x > b^*$, then by the spatial homogeneity and the strong Markov property of X we have

$$v(x) = \mathbb{E}^0 \left[\int_0^{\tau(b^*-x)} e^{-\delta t} U(x + X_t)\, dt \right] + \mathbb{E}^0 \left[e^{-\delta\tau(b^*-x)} U(0)/\delta \right],$$

and also

$$v(x+\xi) = \mathbb{E}^0 \left[\int_0^{\tau(b^*-x)} e^{-\delta t} U(x+\xi+X_t)\, dt \right] + \mathbb{E}^0 \left[e^{-\delta\tau(b^*-x)} v\big(x+\xi+X(\ \tau(b^*-x)\ )\big) \right].$$

Because U is assumed to be strictly increasing, the lower bound (III.2.6) allows us to conclude $v(\cdot)$ increases strictly also.

For the diffusion case, take two independent copies of the process, X and Y, say, and start X at x and Y at y, $x > y > b^*$. Define the coupling time τ by

$$\tau := \inf\{t : X_t = Y_t\}.$$

Now look at

$$v(x) - v(y) = \mathbb{E}\left[\int_0^{\tau} e^{-\delta t}\Big(U(X_t) - U(Y_t)\Big)\, dt; \tau < \tau^Y(b^*)\right]$$
$$+\mathbb{E}\left[\int_0^{\tau^Y(b^*)} e^{-\delta t}\Big(U(X_t) - U(Y_t)\Big)\, dt\right.$$
$$\left. + e^{-\delta\tau^Y(b^*)}\Big(v(X(\,\tau^Y(b^*)\,)) - U(0)/\delta\Big)\ ; \tau \geq \tau^Y(b^*)\right];$$

this expression is strictly positive, again because of the lower bound (III.2.6) and what is assumed of U. ■

The value function in (III.4.2) is best expressed in terms of the resolvent operator of the process X, defined by

$$R_\lambda f(x) := \mathbb{E}^x\left[\int_0^{\infty} e^{-\lambda t} f(X_t)dt\right]$$

for $\lambda > 0$ and for functions f such that the displayed expression is finite. Given any $a \in \mathbb{R}$, an elementary application of the strong Markov Property of X at the time τ_a gives us the identity[14]

$$\begin{aligned} R_\lambda f(x) &= \mathbb{E}^x\left[\int_0^{\tau_a} e^{-\lambda t} f(X_t)\, dt\right] + \mathbb{E}^x\left[e^{-\lambda\tau_a} R_\lambda f(X(\tau_a))\right] \\ &=: {}_aR_\lambda f(x) + \mathbb{E}^x\left[e^{-\lambda\tau_a} R_\lambda f(X(\tau_a))\right] \end{aligned} \tag{III.4.15}$$

relating the resolvent R_λ of X to the resolvent ${}_aR_\lambda$ of X killed the first time it enters $(-\infty, a)$.

For the special choice $f \equiv U$ the resolvent operates such that we can write explicitly

$$R_\delta U(x) = \frac{1}{\delta} U(x)\mathbb{E}^0[e^{-\Gamma X_T}] = \frac{1}{\delta - \psi(-\Gamma)} U(x), \tag{III.4.16}$$

in terms of the Lévy exponent ψ. The finiteness of (III.4.16) for all x is guaranteed if

$$\delta > \psi(-\Gamma), \tag{III.4.17}$$

[14]... often referred to also as Dynkin's formula. See, for instance, Rogers and Williams (2000, III.10).

an algebraic condition equivalent to insisting on the first assumption in (III.2.5). Substituting (III.4.16) into (III.4.15) gives us that

$$ {}_aR_\delta U(x) = \frac{U(x)}{\delta - \psi(-\Gamma)}\Big\{1 - \mathbb{E}^0\Big[\exp\big(-\delta\tau_{a-x} - \Gamma X(\tau_{a-x})\big)\Big]\Big\}. \qquad \text{(III.4.18)} $$

If we fix $b = a$ on the right side of (III.4.2), the expected integral there is of the same form as that in (III.4.15), so that

$$ \begin{aligned} J(x;\tilde{a}) : &= \mathbb{E}^x\Big[\int_0^{\tau_a} e^{-\delta t}U(X_t)\,dt + e^{-\delta\tau_a}U(0)/\delta\Big] \\ &= {}_aR_\delta U(x) + \mathbb{E}^0\big[e^{-\delta\tau(\tilde{a})}\big]U(0)/\delta \\ &= \frac{U(x)}{\delta - \psi(-\Gamma)}\Big\{1 - \mathbb{E}^0\Big[\exp\big(-\delta\tau(\tilde{a}) - \Gamma X(\,\tau(\tilde{a})\,)\big)\Big]\Big\} \\ &\quad + \mathbb{E}^0\big[e^{-\delta\tau(\tilde{a})}\big]U(0)/\delta, \end{aligned} \qquad \text{(III.4.19)} $$

with $\tilde{a} \equiv a - x$. We shall refer often to expressions of the form of (III.4.18), which as is evident from (III.4.19) is the value accumulated by starting X at x and waiting until it enters $(-\infty, a)$.

In principle, of course, we have $v(x,1) = J(x; b^* - x)$, but how explicit this is will depend on how much we know about the joint law of time and place of first entry of X into an interval of form $(-\infty, a)$.

In general, all we can get is the Laplace Transform of (III.4.19) in the $\tilde{a}$-variable, expressed in terms of the Wiener-Hopf factor ψ^- of X. To do this, we employ the fluctuation identity

$$ \int_{-\infty}^0 \theta e^{\theta\xi}\,\mathbb{E}^0\big[\exp\big(-\delta\tau_\xi + \eta X(\tau_\xi)\big)\big]\,d\xi = 1 - \frac{\psi^-(\theta+\eta)}{\psi^-(\eta)}, \qquad \text{(III.4.20)} $$

valid for $\theta > 0$ and all η such that $\psi^-(\eta) < \infty$. Alili and Kyprianou (2004) give a simple derivation assuming $\eta \geq 0$, which we re-work in Appendix A.3 to show that the less restrictive condition on η is enough.

Taking Laplace tranforms in (III.4.19) and using this identity we obtain[15]

[15]Our standing assumption that $\delta > \psi(-\Gamma)$ entails that $\psi(-\Gamma)$ is well-defined, and hence that $\psi^-(-\Gamma) < \infty$.

Proposition III.4.4. *If X is a Lévy Process such that $\psi^-(-\Gamma) < \infty$, then*

$$\begin{aligned}
\mathcal{L}(x;\theta) &\equiv \int_{-\infty}^{0} J(x;\tilde{a})\ \theta e^{\theta\tilde{a}}\ d\tilde{a} \\
&= \frac{U(x)}{\delta - \psi(-\Gamma)}\left(\frac{\psi^-(\theta-\Gamma)}{\psi^-(-\Gamma)}\right) + \frac{U(0)}{\delta}\left(1-\psi^-(\theta)\right)
\end{aligned} \tag{III.4.21}$$

where $\psi^-(z)$ is the Wiener-Hopf factor as it appears in (III.4.7).

From Proposition III.4.4, the value $v(x,1)$ can be computed if the Wiener-Hopf factor ψ^- is nice enough to allow inversion of $\mathcal{L}(x;\cdot)$. This proposition holds even for processes X that are not spectrally negative, but of course even getting hold of the factor ψ^- then becomes a problem.

Without additional structure on X, the value function whose Laplace Transform is (III.4.21) cannot be written in closed form, but there are at least two cases when it can be. Obviously, if X is continuous (and therefore a multiple of drifting Brownian Motion), then (III.4.19) involves only Laplace Transforms of hitting times of levels and can therefore be given explicitly; this is presented in Section III.4.3. If X is a (spectrally negative) compound Poisson process (CPP) of negative exponential jumps added to a drift, then the Laplace Transform (III.4.21) can also be inverted explicitly, and we treat this case in Section III.4.4.

III.4.3. Special case I: Brownian Motion

The form of the solution when X is Brownian Motion is easily deduced from the analysis in Sections III.4.1 and III.4.2. The Lévy exponent for this process is

$$\psi_{BM}(z) = \frac{1}{2}\sigma^2 z^2 + \mu z, \tag{III.4.22}$$

where μ is the drift and $\sigma > 0$ the volatility coefficient. For this process, the factorization (III.4.7) says that for some numbers $\beta^* > 0$, $\alpha^* > 0$,

$$\frac{\delta}{\delta - \psi_{BM}(z)} = \frac{\beta^*\alpha^*}{(\beta^* - z)(\alpha^* + z)}.$$

In other words, $\beta^* > 0$ and $-\alpha^* < 0$ are the roots of the quadratic

$$\psi_{BM}(z) - \delta =: Q_{BM}(z) = \frac{1}{2}\sigma^2 z^2 + \mu z - \delta. \tag{III.4.23}$$

Notice that (III.4.17) can be written in terms of α^* as

$$\Gamma < \alpha^*; \tag{III.4.24}$$

also in terms of α^*, we have the Laplace Transform

$$\mathbb{E}^x\left[e^{-\delta\tau_a}\right] = e^{-\alpha^*(x-a)}, \tag{III.4.25}$$

a well-known fact that can be readily derived from Section III.4.7 below. We can now write down

Proposition III.4.5. *If X is a Brownian Motion with Lévy exponent (III.4.22), then the optimal critical level b^* attaining (III.4.2) satisfies*

$$e^{-\Gamma b^*} = \frac{\Gamma + \beta^*}{\beta^*}, \tag{III.4.26}$$

where $\beta^* > 0$ is the positive root of the quadratic defined in (III.4.23).

Further, by using (III.4.25) in the general expression (III.4.19), and evaluating at $\tilde{a} = b^* - x$, we have

Proposition III.4.6. *If X is a Brownian Motion with Lévy exponent (III.4.22), then the value function (III.4.2) is given by*

$$v(x,1) = \frac{U(b^*)}{\delta - \psi_{BM}(-\Gamma)}\left\{e^{-\Gamma(x-b^*)} - e^{-\alpha^*(x-b^*)}\right\} + e^{-\alpha^*(x-b^*)}U(0)/\delta, \tag{III.4.27}$$

if $x \geq b^*$, and $v(x,1) = U(0)/\delta$ otherwise, the exercise boundary b^* being that given in (III.4.26).

REMARK III.4.7. For $x > b^*$, the value function that we have computed in (III.4.27) satisfies the second order ODE represented by the first term in braces in (III.4.14). Indeed, we see that $v(x) \equiv v(x,1)$ takes the form

$$v(x) = Ae^{-\alpha^* x} + Be^{-\Gamma x}$$

for some constants A and B. The constant B is fixed from requiring that $Be^{-\Gamma x}$ satisfy the non-homogeneous ODE, while A is determined from the continuity of v at b^*: $\lim_{z\downarrow b^*} v(z) = U(0)/\delta$. It turns out in this case that optimality of b^* is equivalent to the smooth pasting condition $\lim_{z\downarrow b^*} v'(z) = 0$.

III.4.4. Special case II: Compound Poisson Process

The exercise boundary for a spectrally negative Lévy Process has been found in Proposition III.4.1. Suppose now that the jump component of X is a compound poisson process (CPP) of exponential negative jumps and that there is no Brownian component. The Lévy exponent of X then takes the form

$$\psi_{CPP}(z) = \mu z - \frac{az}{c+z}, \tag{III.4.28}$$

where $\mu > 0$[16] is a positive drift, $a > 0$ is the rate of arrival of the jumps, and $-c^{-1} < 0$ is the mean of the (exponentially distributed) jump size.

It is now easy to see that the characterization (III.4.11) for β^* is equivalent to β^* being the positive root of the quadratic

$$Q_{CPP} : z \mapsto \left\{\mu z^2 + [\mu c - (a+\delta)]z - c\delta\right\}, \tag{III.4.29}$$

in terms of which we have

$$\psi_{CPP}(z) - \delta = Q_{CPP}(z)/(c+z).$$

Note that $Q_{CPP}(-c) = ac > 0$ implies $c > \alpha^*$, so the condition $\alpha^* > \Gamma$ again ensures the relevant finiteness condition in (III.2.5).

From III.4.1 we quickly deduce

Proposition III.4.8. *Let X have Lévy exponent ψ as in (III.4.28). Then the exercise boundary b^* for the objective (III.4.2) satisfies*

$$e^{-\Gamma b^*} = \frac{\Gamma + \beta^*}{\beta^*}, \tag{III.4.30}$$

where $\beta^* > 0$ is the positive root of the quadratic in (III.4.29).

Consider now the joint law appearing in (III.4.19). The assumption that X has exponential jumps implies that for any $a \in \mathbb{R}$, the time τ_a when X first enters $(-\infty, a)$ is *independent* of $X(\tau_a)$, which because of the memoryless property has itself a shifted exponential distribution. All that is left to contend with, then, are

[16] If $\mu < 0$, then $-X$ is a subordinator, whence $\overline{X}_T \equiv \overline{X}_0$ and $b^* = 0$.

terms of the form $\mathbb{E}^0[e^{-\delta\tau_a}]$, $a < 0$; in fact, we have

$$\mathbb{E}^x[e^{-\delta\tau_a}] = \frac{c-\alpha^*}{c} e^{-\alpha^*(x-a)}, \quad x \geq a, \tag{III.4.31}$$

where $-\alpha^* < 0$ is the negative root of the polynomial Q_{CPP} defined in (III.4.29). The equality (III.4.31) can be verified by checking that its Laplace transform is given correctly by (III.4.20). Notice also that the time of exit from a into $(-\infty, a)$ is not identically 0.

Exploiting the independence just mentioned and employing the Laplace Transform (III.4.31) to compute the expectation in (III.4.19) gives us explicitly the value of the problem in this case. The proof makes clear that optimality of b^* is in this case equivalent to continuity of the value function at b^*.

Proposition III.4.9. *Let X be a CPP with the Lévy exponent ψ_{CPP} in (III.4.28). The value (III.4.2), attained by taking b^* as in Proposition III.4.8, is*

$$\begin{aligned} v(x,1) &= \frac{U(b^*)}{\delta - \psi_{CPP}(-\Gamma)} \left\{ e^{-\Gamma(x-b^*)} - \frac{c-\alpha^*}{c-\Gamma} e^{-\alpha^*(x-b^*)} \right\} + \frac{c-\alpha^*}{c} e^{-\alpha^*(x-b^*)} U(0)/\delta \\ &= \frac{e^{-\Gamma b^*}(c-\Gamma)}{Q_{CPP}(-\Gamma)} \left\{ e^{-\Gamma(x-b^*)} - e^{-\alpha^*(x-b^*)} \right\} + e^{-\alpha^*(x-b^*)} U(0)/\delta, \end{aligned} \tag{III.4.32}$$

for $x \geq b$, and $v(x,1) = U(0)/\delta$ otherwise.

Proof. The first equality follows directly from (III.4.19). The second equality will be true if, and only if,

$$\frac{(c-\Gamma)U(b^*)}{Q_{CPP}(-\Gamma)} \left(\frac{c-\alpha^*}{c-\Gamma} - 1 \right) = \frac{U(0)}{\delta} \left(1 - \frac{c-\alpha^*}{c} \right)$$

which after some algebra involving the relation $\mu\alpha\beta = c\delta$ between the roots of the polynomial $Q_{CPP}(\cdot)$ boils down to checking that we have

$$U(b^*) = U(0) \frac{\Gamma + \beta^*}{\beta^*},$$

which is nothing but the characterization of b^*. Finally, the assertion that $v(x,1) = U(0)/\delta$ for $x < b^*$ is simply the definition of b^*. ■

REMARK III.4.10. The value function we have computed in Proposition III.4.9

is of the same form

$$\tilde{A}e^{-\alpha^* x} + \tilde{B}e^{-\Gamma x}$$

as in the Brownian case. From what was said in Remark III.4.2, if b^* is the exercise boundary specified from (III.4.12), then the value function, $v(x) \equiv v(x,1)$ satisfies

$$(\mathcal{G} - \delta)v(x) = e^{-\Gamma x}, \qquad \text{for } x > b^*, \tag{III.4.33}$$

and $v(x) = U(0)/\delta$ for $x < b^*$. Now the infinitesimal generator, $\mathcal{G}$, of X acts on functions $f \in C^1$ so that

$$\mathcal{G}f(x) = \mu f'(x) + \int_{-\infty}^{0} [f(x+y) - f(x)]a\, c\, e^{cy}\, dy. \tag{III.4.34}$$

Inserting this in (III.4.33), remembering that $v(x) = U(0)/\delta$ for $x < b^*$, the equation (III.4.33) becomes

$$\Big[\mu v' - (a+\delta)v\Big] + \frac{U(0)}{\delta} a e^{c(b^*-x)} - e^{-cx}\int_{b^*}^{x} v(t) ace^{ct} dt = e^{-\Gamma x},$$

whence

$$e^{cx}[\mu v'(x) - (a+\delta)v(x)] = e^{(c-\Gamma)x} - \frac{U(0)}{\delta} a e^{cb^*} - \int_{b^*}^{x} v(t) ace^{ct} dt. \tag{III.4.35}$$

If we differentiate (III.4.35) with respect to x we see that v has to satisfy

$$\mu v''(x) + [c\mu - (a+\delta)]v'(x) - c\delta v(x) = (c-\Gamma)e^{-\Gamma x}, \qquad x > b^*, \tag{III.4.36}$$

a second order ODE whose general solution takes a form in accordance with what we have found. In fact, insisting on the continuity condition at b^*, $v(b^*) = U(0)/\delta$, forces v to take the form expressed in the second equality in (III.4.32).

What is different here from the Brownian situation of the previous Section ? If in (III.4.35) we take $x \downarrow b^*$, and use $\lim_{z\downarrow b^*} v(z) = U(0)/\delta$ we get that

$$e^{-\Gamma b^*} = 1 + \mu \lim_{z\downarrow b^*} v'(z) =: 1 + \mu v'(b^*_+). \tag{III.4.37}$$

The implication of this is that the right derivative of v at b^* is now tied to the requirement that v be continuous at b^*. Indeed, from the representation (III.4.32), equality (III.4.37) forces

$$v'(b^*_+) = \Gamma/(\beta^* \mu); \qquad e^{-\Gamma b^*} = (\beta^*)^{-1}(\Gamma + \beta^*), \tag{III.4.38}$$

showing that the smooth pasting condition fails to hold in this example, and correctly exhibiting β^* as the parameter characterizing the exponential law of $\overline{X}_T$.

Smooth pasting

Alili and Kyprianou (2004) prove that in pricing a perpetual American put on an asset with Lévy-process dynamics, the smooth pasting condition holds at the exercise boundary if, and only if, the point 0 is regular for $(-\infty, 0)$. They conjecture that a similar equivalence holds for optimal stopping problems in general.

For our example, X is a CPP of negative jumps added to a positive drift, and therefore has finite variation. These properties of X make 0 non-regular for $(-\infty, 0)$, and in fact (III.4.38) shows that smooth pasting does not occur. More generally, substituting the expression (III.4.8) for b^* into (III.4.6), we find for each $\varepsilon > 0$ that

$$\begin{aligned}
\delta v(b^* + \varepsilon) &= \mathbb{E}^0\left[1 - e^{-\Gamma(\underline{X}_T + \varepsilon)};\ \underline{X}_T \geq -\varepsilon\right] - 1 \\
&= \mathbb{E}^0\left[\Gamma(\underline{X}_T + \varepsilon);\ \underline{X}_T \geq -\varepsilon\right] - 1 + O(\varepsilon^2) \\
&= \mathbb{E}^0\left[\Gamma\varepsilon;\ \underline{X}_T \geq -\varepsilon\right] + \Gamma\mathbb{E}^0\left[\underline{X}_T;\ \underline{X}_T \geq -\varepsilon\right] - 1 + O(\varepsilon^2) \\
&= \Gamma\varepsilon\, \mathbb{P}^0\left(\underline{X}_T \geq -\varepsilon\right) + \Gamma\ \mathbb{E}^0\left[\underline{X}_T;\ 0 > \underline{X}_T > -\varepsilon\right] - 1 + O(\varepsilon^2),
\end{aligned} \tag{III.4.39}$$

which results because the distribution function of $\underline{X}_T$ has an atom at 0. Because this distribution is however continuous in $(-\infty, 0)$ under the probability $\mathbb{P}^0$, when we subtract $\delta v(b^*) = U(0) = -1$ from the expression above, divide by ε, and let $\varepsilon \downarrow 0$, we obtain the right derivative at b^* as

$$\delta v'(b^*_+) = \Gamma\ \mathbb{P}^0[\underline{X}_T = 0], \tag{III.4.40}$$

which shows equivalence between smooth pasting and regularity of 0 for $(-\infty, 0)$.

Because we have (III.4.38) we even know the size of the atom at 0:

$$\mathbb{P}^0[\underline{X}_T = 0] = \delta/(\beta^*\mu) = \alpha^*/c. \tag{III.4.41}$$

REMARK III.4.11. In the limiting cases $a, c \downarrow 0$, the process X is a pure (and, by assumption, positive) drift so that $\beta^* \to \delta/\mu$, and in (III.4.41) it results as it should in this case that $\mathbb{P}^0[\underline{X}_T = 0] = 1$. In the Brownian case, the limit $\beta^* \to \delta/\mu$ obtains when we send $\sigma \downarrow 0$.

III.4.5. The class φ^c

We take up our earlier claim that for the special case when X is drifting Brownian Motion, the maximization

$$v(x,\theta) = \max_{\Phi \in \varphi^c(\theta)} \left[\mathbb{E}^x \int_0^\infty e^{-\delta t} U(\Phi_t X_t)\, dt\right] \tag{III.4.42}$$

over the class φ^c can be carried out explicitly. We shall assume $\Phi(0-) = 1$, and because the control processes $\Phi \in \varphi^c$ can take values in $[0, \Phi(0-)]$, the exercise boundary characterizing the solution will need to be specified for each $\theta \in [0,1]$. Define the process $M \equiv (M_t)_{t\geq 0}$ by

$$M_t = \int_0^t e^{-\delta s} U(\Phi_s X_s) ds + e^{-\delta t} v(X_t, \Phi_t), \tag{III.4.43}$$

analogously to (III.4.13). This process is a martingale under the optimal choice of Φ. If we suppose for a moment that Φ were restricted to have bounded density $|\dot{\Phi}| \leq K$ with respect to Lebesgue measure, an application of Itô's Lemma to M tells us that we need to have

$$\max_{\dot{\Phi} \leq 0} \left\{ U(\Phi x) - (\delta - \mathcal{G}) v(x, \Phi) + v_\theta(x, \Phi)\dot{\Phi} \right\} = 0, \tag{III.4.44}$$

where $v_\theta(x, \cdot)$ is the derivative of v with respect to its second argument.

Thus, it is optimal to take $\dot{\Phi} = 0$ while $v_\theta \geq 0$. Decreasing Φ is called for as soon as $v_\theta < 0$, and until $v_\theta \geq 0$ obtains again. Now because $v(x,\theta)$ is increasing in x, and Φ_t is $\mathcal{F}_t$-measurable, it is not optimal to have $\dot{\Phi}(t) < 0$ at t if $X_t > \underline{X}_t$. The optimal process Φ^* is therefore a singular process of bang-bang type such that if $\Phi_t^* > 0$, we always have $X_t \geq b^*(\Phi_t^*)$ where b^* is some function to be determined.

Suppose that a choice of Φ were prescribed via a sub-optimal function $b : [0,1] \to \mathbb{R}$, whereby Φ is decreased at t if, and only if, $\Phi_t > 0$ and $X_t < b(\Phi_t)$. If $\theta \in [0,1]$ and X starts at $X_0 = x > b(\theta)$, then the value function (III.4.42) is expressible as

$$v(x,\theta) = \max_{b(\cdot)} \mathbb{E}^x \Big[\int_0^{\tau(b(\theta))} e^{-\delta t} U(\theta X_t) dt + e^{-\delta \tau(b(\theta))} v(b(\theta), \theta) \Big], \qquad \text{(III.4.45)}$$

the first term being the value accumulated until time $\tau(b(\theta))$, and the second term being the value from employing the optimal control thereafter.

We can compute explicitly the form of the function b^* attaining (III.4.45).

Proposition III.4.12. *If X is Brownian Motion with Levy exponent (III.4.22), then the optimal process $\Phi^* \in \varphi^c(\theta)$ attaining value (III.4.45) has $\Phi_t^* = 0$ as soon as $\underline{X}_t = -1/\beta^* =: \underline{b}^*$, and otherwise satisfies $X_t \geq b^*(\Phi_t)$, where the increasing function $b^* : [0,1] \mapsto [-(\beta^*)^{-1}, -(\beta^* + \Gamma)^{-1}]$ is given by*

$$b^*(z) = -(\beta^* + \Gamma z)^{-1}. \qquad \text{(III.4.46)}$$

Proof. We have already argued that Φ^* is such that $X_t \geq b^*(\Phi_t^*)$ whenever $\Phi_t^* > 0$; we now need to determine the function $b^*(\cdot)$. Given a possibly suboptimal monotone function $b(\cdot)$, define the function η through $\eta(b(\theta)) = \theta$, and suppose $x = b^*(\theta)$. Then the objective in (III.4.45) is

$$\mathbb{E}^x \Big[\int_0^{\tau_{\underline{b}}} - \exp\big(-\delta t - \Gamma \eta(\underline{X}_t) X_t\big)\, dt - e^{-\delta \tau(\underline{b})}/\delta \Big], \qquad \text{(III.4.47)}$$

where

$$\underline{b} := \inf\{y : \eta(y) > 0\}.$$

We now choose η to maximize the expression displayed above and hence attain $v(b^*(\theta), \theta) \equiv v(x, \eta^*(x))$. Note first that the expected integral here can be computed under the law of a standard Brownian Motion W by invoking a change of measure. Thus,

$$
\begin{aligned}
\mathbb{E}^x\Big[\int_0^{\tau(\underline{b})} &-\exp\big(-\delta t - \Gamma\eta(\underline{X}_t)X_t\big)\, dt\Big] \\
&= -e^{-cx/\sigma}\tilde{\mathbb{E}}^{x/\sigma}\Big[\int_0^{\tau(\underline{b}/\sigma)} \exp\big(-\delta t - \Gamma\eta(\sigma\underline{W}_t)\sigma W_t\big)\exp\big(cW_t - \frac{1}{2}c^2 t\big)\, dt\Big] \\
&= -e^{-cx/\sigma}\tilde{\mathbb{E}}^{x/\sigma}\Big[\int_0^{\tau(\underline{b}/\sigma)} \exp\big(-\kappa t\big)\exp\big(-(\Gamma\sigma\eta(\sigma\underline{W}_t) - c)W_t\big)\, dt\Big] \\
&= -e^{-cx/\sigma}\tilde{\mathbb{E}}^{x/\sigma}\Big[\int_0^{\tau(\underline{b}/\sigma)} \exp\big(-\kappa t\big) \\
&\qquad \times \exp\big(-(\Gamma\sigma\eta(\sigma\underline{W}_t) - c)[\underline{W}_t + (W_t - \underline{W}_t)]\big)\, dt\Big], \qquad \text{(III.4.48)}
\end{aligned}
$$

where $c = \mu/\sigma$, $\kappa = \delta + \frac{1}{2}c^2$, and $\tilde{\mathbb{E}}$ signifies expectation in the law of W.

Now if we write $A_y = \{t : \underline{W}_t = y\}$ for the time-interval on which the infimum process remains at y, then $\{W_s - y, s \in A_y\}$ is independent of y and has the law of a Brownian excursion upwards from the level y. Now write U for the set of excursions of W and $\Xi(\cdot,\cdot)$ for the Poisson random measure of excursions, defined on $\mathbb{R}^+ \times U$, with expectation measure Lebesgue $\times$ n. The Poisson property of Ξ implies in particular[17] that if we fix $s \in \mathbb{R}^+$, then for any measurable function $\phi : U \to \mathbb{R}$ such that $\int_U |\phi(\xi)| n(d\xi) < \infty$, and any $t \geq s$, the process

$$
\begin{aligned}
M_t &:= \int\int_{(s,t]\times U} \phi(\xi)\Xi(dv, d\xi) - \int_{(s,t]} n(\phi)\, dv \\
&\equiv \int\int_{(s,t]\times U} \phi(\xi)\Xi(dv, d\xi) - \int_{(s,t]} \left(\int_U \phi(\xi) n(d\xi)\right) dv
\end{aligned}
$$

is a martingale (with respect to the excursion filtration), indexed of course by the *local time* of W at 0.

Motivated by the fact (Lévy's Theorem; see Appendix A.2) that this local time has the same law as the process $-\underline{W}$, in the expectation (III.4.48) we now make the change of variable

$$
\tau_y = \inf\{t : \underline{W}_t = y\},
$$

[17] See, for example, Rogers (1989).

and invoke the martingale property displayed above to change (III.4.48) into

$$
\begin{aligned}
&- e^{-cx/\sigma}\tilde{\mathbb{E}}^{x/\sigma} \int\int_{(\underline{b}/\sigma, x/\sigma]\times U} \exp\big\{ -\kappa\tau_y - \big(\Gamma\sigma\eta(\sigma y) - c\big)y\big\} \\
&\qquad\qquad \times \left(\int_0^{\zeta(\xi)} e^{-\kappa s - (\Gamma\sigma\eta(\sigma y)-c)|\xi_s|}\, ds\right)\Xi(dy, d\xi) \\
&= -e^{-cx/\sigma}\tilde{\mathbb{E}}^{x/\sigma} \int_{\underline{b}/\sigma}^{x/\sigma} \exp\big\{ -\kappa\tau_y - \big(\Gamma\sigma\eta(\sigma y) - c\big)y\big\} \\
&\qquad\qquad \times n\left(\int_0^{\zeta(\xi)} e^{-\kappa s - (\Gamma\sigma\eta(\sigma y)-c)|\xi_s|}\, ds\right) dy \\
&= -e^{-cx/\sigma}\tilde{\mathbb{E}}^{x/\sigma} \int_{\underline{b}/\sigma}^{x/\sigma} \exp\big\{ -\kappa\tau_y - (\Gamma\sigma\eta(\sigma y) - c)y\big\} \\
&\qquad\qquad \times \lambda(\kappa, \Gamma\sigma\eta(\sigma y) - c)\, dy
\end{aligned} \tag{III.4.49}
$$

where

$$
\begin{aligned}
\lambda(r,\theta) :&= n\left(\int_0^{\zeta(\xi)} e^{-rt-\theta|\xi_t|} dt\right) \\
&\equiv \int_U \int_0^{\zeta(\xi)} e^{-ru-\theta|\xi_u|} du\; n(d\xi) \\
&= 2/(\sqrt{2r} + \theta),
\end{aligned} \tag{III.4.50}
$$

computed in Appendix A.2, is a typical (undiscounted) contribution of an excursion by B above 0. For more detail, refer to Appendix A.2.

Putting everything together, recalling the Laplace Transform $\tilde{\mathbb{E}}^z[e^{-\kappa\tau_y}] = e^{-\sqrt{2\kappa}(z-y)}$ for $z > y$ gives the value in (III.4.47) as

$$
\begin{aligned}
v(x, \eta^*(x)) &= -2e^{-\alpha^* x} \int_{-\infty}^{x/\sigma} \frac{\exp\big(\sqrt{2\kappa} - \Gamma\sigma\eta^*(\sigma y) + c\big)y}{\sqrt{2\kappa} + \Gamma\sigma\eta^*(\sigma y) - c}\, dy \\
&= -2e^{-\alpha^* x} \int_{-\infty}^{x/\sigma} \frac{\exp\big(\alpha^*\sigma - \Gamma\sigma\eta^*(\sigma y)\big)y}{\beta^*\sigma + \Gamma\sigma\eta^*(\sigma y)}\, dy \\
&= -2e^{-\alpha^* x} \int_{-\underline{b}^*/\sigma}^{x/\sigma} \frac{\exp\big(\alpha^*\sigma - \Gamma\sigma\eta^*(\sigma y)\big)y}{\beta^*\sigma + \Gamma\sigma\eta^*(\sigma y)}\, dy \\
&\quad - e^{-\alpha^*(x-\underline{b}^*)}/\delta
\end{aligned} \tag{III.4.51}
$$

where we have used $\sqrt{2\kappa} = \beta^*\sigma + c = \alpha^*\sigma - c$, and where it is a simple exercise to verify that the maximizing $\eta^*(\cdot)$ is

$$\eta^*(y) = \begin{cases} -\frac{1}{\Gamma}(\beta^* + 1/y) & \text{if } y \geq \underline{b}^* = -1/\beta^*, \\ 0 & \text{otherwise} \end{cases} \tag{III.4.52}$$

a function which specifies Φ^* through the recipe $\Phi_t^* = \eta^*(\underline{X}_t)$. Inverting the expression for η^* gives the required expression (III.4.46). ■

By using (III.4.52) in (III.4.51) we obtain

Corollary III.4.13. Under the optimal strategy $\Phi^* \in \varphi^c$ for Brownian Motion with Lévy exponent (III.4.22) we have, with $\theta \in [0, 1]$,

$$\begin{aligned} v(b^*(\theta), \theta) &\equiv v(x, \eta^*(x)) \\ &= \frac{e^{-\alpha^* x}}{\sqrt{2\kappa}} \Big\{ e^{1+2\sqrt{2\kappa}x/\sigma} \Big(\frac{x}{\sigma} - \frac{1}{2\sqrt{2\kappa}} \Big) - e^{1-2\sqrt{2\kappa}/(\beta^*\sigma)} \Big(-\frac{1}{\beta^*\sigma} - \frac{1}{2\sqrt{2\kappa}} \Big) \Big\} \\ &\quad - e^{-\alpha^*(x+1/\beta^*)}/\delta, \end{aligned} \tag{III.4.53}$$

where $\sqrt{2\kappa} = \beta^*\sigma + c = \alpha^*\sigma - c$.

We can now write down the value function (III.4.42).

Proposition III.4.14. *If $\varphi = \varphi^c$ and X is Brownian Motion with Lévy exponent (III.4.22), then for $\theta \in [0, 1]$ we have*

$$v(x, \theta) = \begin{cases} U(0)/\delta, & \text{if } x \leq -1/\beta^* \\ v(x, \eta^*(x)), & \text{if } -1/\beta^* < x \leq b^*(\theta). \end{cases} \tag{III.4.54}$$

If $x > b^*(\theta)$, then

$$v(x, \theta) = \frac{U(\theta b^*(\theta))}{\delta - \psi(-\Gamma\theta)} \Big\{ e^{-\Gamma\theta(x-b^*(\theta))} - e^{-\alpha^*(x-b^*(\theta))} \Big\} + e^{-\alpha^*(x-b^*(\theta))} v(b^*(\theta), \theta), \tag{III.4.55}$$

where $v(b^*(\theta), \theta) \equiv v(\tilde{x}, \eta^*(\tilde{x}))$ is given from (III.4.53).

Proof. The choice of η^* in Proposition (III.4.12) ensures the validity of (III.4.54). To prove (III.4.55), deduce the first term in (III.4.45) from (III.4.18). ■

III.4.6. Comparison of strategies for φ^c and φ^s

We found in Proposition III.4.12 the form of the barrier which when reached induces the agent to adjust his holdings process Φ; given that $\Phi = \theta$, the strategy is to act as soon as $\underline{X}$ hits

$$b_c^*(\theta) := -[\Gamma\theta + \beta^*]^{-1}.$$

On the other hand, if $\theta \in [0,1]$ is the amount of shares available, then by a simple scaling argument in the derivation (which implicitly assumed $\theta = 1$) of the exercise boundary b^* in (III.4.26) it is easy to see that the agent owning only θ shares and restricted to a holdings process $\Phi \in \varphi^s$ would take as his critical level for $\underline{X}$

$$b_s^*(\theta) := -\frac{1}{\Gamma\theta}\log\Big(1 + \frac{\Gamma\theta}{\beta^*}\Big).$$

Because the option to default is worth more to the agent with $\Phi \in \varphi^c$ than it is to the more restricted agent, we would expect the former to have critical levels that are *higher*. Intuitively, the impact of the agent's disposing of a fraction of his share ownership is potentially much less serious than that of disposing of *all* that he owns. Indeed, the validity of

$$b_c^*(\theta) \geq b_s^*(\theta)$$

for all $\theta \in [0,1]$ is equivalent to having

$$e^{\xi} \leq \frac{1}{1-\xi},$$

where $\xi := \frac{\Gamma\theta}{\Gamma\theta + \beta^*} \in [0, \frac{\Gamma}{\Gamma + \beta^*}] \subset [0,1)$. The displayed inequality is strict for $1 \geq \xi > 0$, and we have equality only in the limiting case $\theta \to 0$ ($\xi \to 0$).

III.4.7. Exercise Boundary for a diffusion

What underlies our calculations up to now is (i) the special way in which the resolvent operates on the exponential form assumed for the utility U, and (ii) the availability of the Wiener-Hopf factors for spectrally negative Lévy Processes X. An alternative setting in which the general form of the value function can still be written down is when the process X is a continuous (one-dimensional) diffusion living on $\mathbb{R}$, the reason being that the resolvent operator for such a process has a density expressible in terms of eigenfunctions of the infinitesimal generator; see Rogers and Williams (2000, Section V.50). We carry out in this section the computations for such a diffusion X and a general increasing (utility) function U not necessarily of exponential form.

Let m and s be, respectively, the speed measure and the scale function of the diffusion X. Then the resolvent of X has a density with respect to m so that for each function f such that the displayed quantities exist, we can write the action of the operator as an integral:

$$R_\lambda f(x) = \int_{\mathbb{R}} r_\lambda(x,y) f(y) m(dy).$$

The density $r_\lambda(x,\cdot)$ is expressible as

$$r_\lambda(x,y) = c_\lambda \Psi_\lambda^+(x \wedge y)\Psi_\lambda^-(x \vee y), \qquad x, y \in \mathbb{R}, \tag{III.4.56}$$

where the Wronksian c_λ is defined through

$$2c_\lambda^{-1} = \{\Psi_\lambda^-(z) D_s \Psi_\lambda^+(z) - \Psi_\lambda^+(z) D_s \Psi_\lambda^-(z)\}, \tag{III.4.57}$$

independently of the choice of the point $z \in \mathbb{R}$, and where D_s is the differential operator $(\frac{1}{Ds})D$ with D being differentiation. The functions $\Psi_\lambda^\pm$ [18] satisfy the second-order ODE

$$\mathcal{G}f \equiv \frac{1}{2} D_m D_s f = \lambda f, \tag{III.4.58}$$

the generator $\mathcal{G}$ having been written here in self-adjoint form, with $D_m \equiv (\frac{1}{Dm})D$. We mention also that the Ψ_λ functions are Laplace Transforms of hitting times for X; in particular,

$$\Psi_\lambda^-(x) = \mathbb{E}^x(e^{-\lambda\tau_q}), \qquad \text{for } x > q, \tag{III.4.59}$$

where changing the reference point q will affect Ψ_λ^- only through a scaling. Thus, if $x > y > q$, then by writing $\tau_q \equiv \tau_y + (\tau_q - \tau_y)$ and using the strong Markov

[18] We are here following the exposition and notation in Rogers and Williams (2000). To avoid confusion, we make clear that the functions $\Psi_\lambda^\pm$ appearing here are different from the Wiener-Hopf factors $\psi^\pm$ in (III.4.7).

Property at y, we deduce that

$$\mathbb{E}^x[e^{-\lambda\tau_y}] = \Psi_\lambda^-(x)/\Psi_\lambda^-(y), \tag{III.4.60}$$

irrespective of the choice of $q < y$.

Now the identity (III.4.15),

$$R_\lambda f(x) = {}_aR_\lambda f(x) + \mathbb{E}^x\left[e^{-\lambda\tau_a}R_\lambda f(X(\tau_a))\right],$$

which we obtained in Section III.4.2, is of course no less valid when X is a diffusion than when X is a Lévy Process as in that Section. In fact, because X is continuous, the identity simplifies to yield

$${}_aR_\lambda f(x) = R_\lambda f(x) - \mathbb{E}^x\left[e^{-\lambda\tau_a}\right]R_\lambda f(a). \tag{III.4.61}$$

From this and (III.4.56) we obtain the density of the killed resolvent ${}_aR_\lambda$ as

$$\begin{aligned} {}_ar_\lambda(x,y) &= r_\lambda(x,y) - \frac{\Psi_\lambda^-(x)}{\Psi_\lambda^-(a)}r_\lambda(a,y) \\ &= c_\lambda\left\{\Psi_\lambda^+(x\wedge y)\Psi_\lambda^-(x\vee y) - \frac{\Psi_\lambda^-(x)}{\Psi_\lambda^-(a)}\Psi_\lambda^+(a)\Psi_\lambda^-(y)\right\}, \quad x,y>a, \end{aligned} \tag{III.4.62}$$

where we have used (III.4.60)[19].

We are now able to write down an integral expression for the value (III.4.2); this is

$$\begin{aligned} v(x) &= \max_b\left\{\int_b^\infty {}_br_\delta(x,y)U(y)m(dy) + \mathbb{E}^x\left(e^{-\delta\tau_b}\right)U(0)/\delta\right\} \\ &= \int_{b^*}^\infty {}_{b^*}r_\delta(x,y)U(y)m(dy) + \frac{\Psi_\delta^-(x)}{\Psi_\delta^-(b^*)}U(0)/\delta, \end{aligned} \tag{III.4.63}$$

Our aim is to now find the value b^* that attains (III.4.63), which we do in

Proposition III.4.15. *Assume X is a diffusion on $\mathbb{R}$, U is increasing and finite*

[19]Analytically, $r_\lambda(\cdot,\cdot)$ is of course nothing but the free-space Green's function, and ${}_ar_\lambda(\cdot,\cdot)$ that for the interval $[a,\infty)$, pertaining to the differential operator $\mathcal{G}$. This implies that the killed resolvent must satisfy ${}_ar_\lambda(a,\cdot)\equiv 0$ and $\mathcal{G}\ {}_ar_\lambda(x,y)=0$ for each $x>a$, $x\neq y$, so that (III.4.60) can be written also as $\mathbb{E}^x[e^{-\lambda\tau_a}] = \frac{r_\lambda(x,a)}{r_\lambda(a,a)}$.

on $\mathbb{R}$, and that $\varphi = \varphi^s$. If b^* satisfies the integral equation

$$\int_b^\infty \Psi_\delta^-(y)\big[U(y) - U(0)\big]m(dy) = 0, \tag{III.4.64}$$

then b^* attains (III.4.63) and is uniquely determined. If (III.4.64) holds for no b, then $b^* = -\infty$.

Proof. Write $v_b(x)$ for the term in braces being maximized in (III.4.63), and let $g(b) := \partial v_b(x)/\partial b$. Differentiating $v_b(x)$ in b we get

$$\begin{aligned}
g(b) &= -\frac{\Psi_\delta^-(x)}{\Psi_\delta^-(b)^2}\Bigg\{\int_b^\infty U(y)\Psi_\delta^-(y)c_\delta\Big(\Psi_\delta^-(b)D\Psi_\delta^+(b) - \Psi_\delta^+(b)D\Psi_\delta^-(b)\Big)\ m(dy) \\
&\qquad + \frac{U(0)}{\delta}D\Psi_\delta^-(b)\Bigg\} \\
&= -2\,\frac{\Psi_\delta^-(x)}{\Psi_\delta^-(b)^2}(Ds(b))\Bigg\{\int_b^\infty\ U(y)\Psi_\delta^-(y)m(dy) \\
&\qquad + \frac{U(0)}{\delta}\frac{1}{2}D_s\Psi_\delta^-(b)\Bigg\} \\
&= -2\,\frac{\Psi_\delta^-(x)}{\Psi_\delta^-(b)^2}(Ds(b))\Bigg\{\int_b^\infty\ U(y)\Psi_\delta^-(y)m(dy) \\
&\qquad - \frac{U(0)}{\delta}\int_b^\infty \frac{1}{2}D_mD_s\Psi_\delta^-(y)m(dy)\Bigg\} \\
&= -2\,\frac{\Psi_\delta^-(x)}{\Psi_\delta^-(b)^2}(Ds(b))\Bigg\{\int_b^\infty\ \Big(U(y) - U(0)\Big)\Psi_\delta^-(y)m(dy)\Bigg\},
\end{aligned}$$

where we have used the definition of Ψ_δ^- as a function satisfying (III.4.58) (with λ replaced by δ).

If b^* satisfies the integral equation (III.4.64), then $g(b^*) = 0$, whence b^* is a stationary point for $v_b(x)$. But because the scale function s and the utility function U are increasing, and $\Psi_\delta^-(\cdot)$ is non-negative, we deduce that $g(0) < 0$, $b^* < 0$, and that the stationary point is indeed a local maximum. But this local maximum must in fact be the global maximum, because $g(\cdot)$ can have at most one zero.

The only other possibility is that $g(b) < 0$ for all $b \leq 0$, and in that case we must have $b^* = -\infty$. ■

If $X_t = \sigma W_t + \mu t$ is the Brownian Motion of Section III.4.3, then the densities s' and m' with respect to Lebesgue measure of the scale and speed of X are determined through

$$(m'(x)\ s'(x))^{-1} = \sigma^2; \qquad s'(x) = e^{-2\mu x/\sigma^2}.$$

It is straightforward to verify that the functions $\Psi^{\pm}$ satisfying (III.4.58) are, to within unimportant multiplicative constants,

$$\Psi_\delta^-(x) = e^{-\alpha^* x}; \qquad \Psi_\delta^+(x) = e^{\beta^* x}, \tag{III.4.65}$$

where $-\alpha^* < 0$, $\beta^* > 0$ are the roots of the polynomial (III.4.23). Compare with (III.4.25). It can be verified directly that the number b^* solving (III.4.64) in this case is the same b^* given in Proposition III.4.5.

III.5. The continuous-time problem with Lévy output: the two-firm case

Following on from the analysis of Section III.4, we now present the continuous-time model of that section with more than one firm. We shall consider special examples of bivariate Lévy processes $X = (X_1, X_2)^T$. Closed-form solutions to (III.2.4) are of course no longer available, and for ease of exposition and for numerical computations, we restrict the number of firms to $N = 2$. No new notions are involved for larger N, but numerical implementation is bound to become much more intricate.

We assume that $\varphi = \varphi^s(\mathbf{1})$, $\mathbf{1} := (1,1)^T$, in (III.2.4). The decision to default one or other or both of the firms reduces the model to that studied in Section III.4, so from what we did there the decision to default entails value

$$\Psi(x, \mathbf{1}) := v(I_1 \cdot x, I_1) \vee v(I_2 \cdot x, I_2) \tag{III.5.1}$$

that can be written down explicitly, where $I_k \equiv I_{\{k\}}$ is the indicator vector for firm k.

It might of course be optimal to hesitate to default, so the optimality equation for the two-firm problem has the form

$$v(x, \mathbf{1}) = \max_{\mathbb{S}} \mathbb{E}^x \Big[\int_0^{\tau(\mathbb{S})} e^{-\delta t} U(\mathbf{1} \cdot X_t) dt + e^{-\delta \tau(\mathbb{S})} \Psi(X(\tau(\mathbb{S})), \mathbf{1}) \Big], \qquad \text{(III.5.2)}$$

where $\tau(\mathbb{S}) := \inf\{t : X_t \in \mathbb{S}\}$ is the first entry time into $\mathbb{S} \subseteq \mathbb{R}^2$, and where we need to solve for the stopping set $\mathbb{S}^*$[20] attaining (III.5.2).

III.5.1. Formulation as a free-boundary problem

The martingale optimality principle that was used in Remark III.4.2 can be applied in exactly the same way here, giving a complementarity problem of the same form for $v(x, \mathbf{1})$:

$$\min_{\mathbb{C}} \{(\delta - \mathcal{G}) v(x, \mathbf{1}) - U(\mathbf{1} \cdot x), v(x, \mathbf{1}) - \Psi(x)\} = 0, \qquad \text{(III.5.3)}$$

where we recall $\mathcal{G}$ is the infinitesimal generator for the process X. The continuation set $\mathbb{C}^*$ (and hence $\mathbb{S}^*$) is defined from insisting that the minimand here be the first term in braces if, and only if, $x \in \text{int}(\mathbb{C}^*)$.

If X is Brownian motion, (III.5.3) is a free-boundary problem for an elliptic PDE, to which numerical schemes can be applied. Problems of similar kind typically arise in the pricing of financial derivatives or traded assets with features of early exercise - for instance, American Options on several assets, or convertible bonds. Even in cases when the payoff function for such contracts helps in choosing adequate boundary conditions to use at the edges of a truncated grid in a numerical scheme, the problem remains notoriously difficult. From a PDE point of view, implementing conditions at the free boundary is a very non-trivial task; see, for example, Bermudez and Nogueiras (2003). Our difficulties are compounded by the facts that (i) there is no terminal ('expiry') time in our problem, (ii) we know very little about the behaviour of the value function at the edges of any truncated grid we care to choose. If X has jumps, $\mathcal{G}$ is an integro-differential operator, which makes things harder still.

The traditional approach to solving free-boundary problems arising in finance is to approximate the pricing PDE by a finite-difference equation and to then solve the latter by an iterative procedure. Features of early exercise can be dealt

[20] As in the one dimensional case, we maintain the convention that $\mathbb{S}^*$ is an open set, so boundary points of $\mathbb{S}^*$ are assumed to be in $\mathbb{C}^* = \mathbb{R}^2 \setminus \mathbb{S}^*$.

with at each step of the iteration by replacing the value function with the exercise value if the latter is larger. Kwok (1998, Sec. 5.3) contains a description of the method, commonly known as PSOR, in the context of pricing American options. Dempster and Hutton (1999) approach the same problem by recasting the variational inequality as a linear program. Pricing problems with early exercise, when discretized, become complementarity problems. Cottle, Pang and Stone (2003) present several algorithms for solving such problems when the operator involved is linear. Oberman (2003) presents a class of finite difference schemes for solving nonlinear elliptic and parabolic PDE's which he applies to several free-boundary problems, including one arising from utility-indifference pricing in incomplete markets; see also Oberman and Zariphopoulou (2003). Bermudez and Nogueiras (2003) and Bermudez and Webber (2003) solve a discretized weak formulation of the pricing parabolic differential equation arising in the contexts of convertible bond pricing and Amerasian options; their method benefits from employing finite element rather than finite difference schemes. PDE-based methods such as those just mentioned become infeasible in high dimensions, which is why there has recently been much interest in simulation-based methods. For example, Bally and Pagès (2003) approximate the underlying process by a simulated Markov Chain such that each state of the chain corresponds to a subset of a partition of the state space of the original process. Berridge (2004) employs a similar approach, approximating the infinitesimal generator of the underlying process by employing Kushner and Dupuis' (2001) local-consistency conditions on a grid of simulated points. Rogers (2002) attacks the dual form of the American option pricing problem, by representing the Snell envelope that characterizes the solution as a minimization over a space of martingales.

The approach we take to our particular infinite-horizon problem is to approximate the underlying process X by a finite state-space Markov chain and then solve the finite dynamic program that results. The impact of this is that finite-difference matrices now correspond to probability transition matrices that immediately satisfy stability and convergence criteria. For more details on such an approach, see Kushner and Dupuis (2001). Nelson and Ramaswamy (1990) present a binomial method for approximating SDE's with continuous drift and volatility coefficients. The key to their method is to map the given process to one of constant volatility, to which binomial schemes on a *regular* grid can be applied.

III.5.2. Occurrence of default

The defaults that concern us in this two-firm model are those which occur when the share amount $\Phi_t = \mathbf{1}$; otherwise we are back to the one-firm scenario of Section III.4.

As explained at the end of Section III.3, one particular motivation for solving a two-firm model in continuous-time was to understand whether the contagion region can be non-empty. In the present context, and using similar notation to that of Section III.3, this region is defined by

$$\mathbb{D}^* := \{x \ : \ v(x,\mathbf{1}) > \Psi(x,\mathbf{1}),\ \Psi(x,\mathbf{1}) = v(x,\mathbf{0}) = U(0)/\delta)\}. \qquad \text{(III.5.4)}$$

If X is continuous, then we would see the share amount change from $\Phi = \mathbf{1}$ to $\Phi = \mathbf{0}$, and both firms therefore being defaulted at once, as X enters $\mathbb{S}^*$ from $\mathbb{D}^*$.

We do not know of simple conditions that ensure $\mathbb{D}^*$ is not the empty set. One sufficient condition can be deduced by considering a class $\varphi^\sharp \subset \varphi^s$ of suboptimal rules, as follows. Suppose that decision to default is based on observing the process

$$Y^m := X \cdot (1,m)^T,$$

where $m \in \mathbb{R}$ is a free parameter, and suppose further that the only action possible at time of default is to change Φ from $\mathbf{1}$ to $\mathbf{0}$, that is, to default both firms at the same time. When specialized in this way, the problem becomes one-dimensional. If Y^m is a Lévy Process or a one dimensional diffusion on $\mathbb{R}$, the optimal strategy $\Phi^\sharp \in \varphi^\sharp$ based on Y^m is characterized by a continuation set $\mathbb{C}^\sharp = [b^\sharp, \infty)$ such that for each $t \in \mathcal{T}$, $\Phi^\sharp_t = \mathbf{1}$ if, and only if, $\underline{Y}^m(t) \geq b^\sharp$. In the context of the two-dimensional problem, this implies a contagion region

$$\mathbb{D}^\sharp := \{x : \ x \cdot (1,m)^T > b^\sharp,\ x_1 < b_1^*,\ x_2 < b_2^*\},$$

a triangular region in the plane. We can now formulate

Proposition III.5.1 (Linear Strategies). *If $Y^m := X \cdot (1,m)^T$ is a Lévy Process or a diffusion on $\mathbb{R}$, then $\mathbb{D}^* \supseteq \mathbb{D}^\sharp$. This statement is true for each $m \in \mathbb{R}$.*

Proof. If $\mathbb{D}^\sharp = \emptyset$ then there is nothing to prove. Otherwise, denote by $v^\sharp(x,\mathbf{1})$ the value attainable by restricting admissible strategies to $\varphi^\sharp$, and suppose $x \in \mathbb{D}^\sharp$, whence $v^\sharp(x, \mathbf{1}) > U(0)/\delta$. But because $x_1 < b_1^*$ and $x_2 < b_2^*$, we have $\Psi(x, \mathbf{1}) = U(0)/\delta$ by definition of the b_i, so if it were true that $x \notin \mathbb{D}^*$, we would have $v(x, \mathbf{1}) = \Psi(x, \mathbf{1}) < v^\sharp(x, \mathbf{1})$, contradicting the fact that the value attained in $\varphi^\sharp$ cannot exceed that of the optimal rule $\Phi^* \in \varphi^s$. ■

As a corollary, we deduce

Corollary III.5.2. If $b^\sharp < b_1^* + mb_2^*$, then $\mathbb{D}^* \neq \emptyset$.

Using the characterization in (III.4.64), we investigated numerically whether the condition in Corollary III.5.2 holds for different choices of the utility function U and taking X_1, X_2 to be drifting Brownian Motions. Our results were negative for several forms of utility functions that we tried, including piecewise linear, piecewise CRRA, and (piecewise) exponential.

III.5.3. Numerical solution

In order to solve numerically for the value function (III.5.2), we discretize the problem by approximating the underlying process X by a discrete-time Markov Chain $\tilde{X}$ with a *finite* state-space. A dynamic programming equation results which can in principle be solved exactly in finite time. Here we describe algorithms for doing this.

Define the regular grid of points

$$\mathcal{Z}_M \equiv \{(x_i, y_j)^T = (x_0 + ih_1, y_0 + jh_2)^T : 0 \leq i \leq M_1 - 1, \\ 0 \leq j \leq M_2 - 1\}, \qquad \text{(III.5.5)}$$

where $h = (h_1, h_2)^T$ is the vector of grid spacings and $M \equiv (M_1, M_2)^T$ specifies the number of points in each of the coordinate directions. We write

$$\partial\mathcal{Z}_N \equiv \{(x_i, y_j)^T : \ i \in \{0, M_1 - 1\} \text{ or } j \in \{0, M_2 - 1\}\} \qquad \text{(III.5.6)}$$

for the boundary of the grid $\mathcal{Z}_N$. Now let $\tilde{X} := \{\tilde{X}_n, n \geq 0\}$ be a Markov Chain with state space $\mathcal{Z}$ and transition matrix $\tilde{P}$, and let $\Delta \equiv \Delta(h) > 0$.

Define also the value function

$$\tilde{v}(z, \mathbf{1}) := \max_{\tilde{\mathbb{S}}} \tilde{\mathbb{E}}^z \Big[\sum_{n=0}^{\tilde{\tau}-1} \tilde{\beta}^n U(\mathbf{1} \cdot \tilde{X})\Delta + \tilde{\beta}^{\tilde{\tau}} \Psi(\tilde{X}(\tilde{\tau}), \mathbf{1})\Big], \quad z \in \mathcal{Z}, \qquad \text{(III.5.7)}$$

where $\tilde{\beta} \in (0, 1)$ is a discount factor, $\tilde{\mathbb{S}} \subseteq \mathcal{Z}$ and $\tilde{\tau} \equiv \tilde{\tau}(\tilde{\mathbb{S}})$ is the time of first entry into $\tilde{\mathbb{S}}$. From the dynamic programming principle, the function (III.5.7) solves the system of equations

$$\tilde{v}(z, \mathbf{1}) = \max\left\{U(\mathbf{1} \cdot \tilde{X})\Delta + \tilde{\beta} \sum_{y \in \mathcal{Z}} \tilde{p}_{zy}\, \tilde{v}(y, \mathbf{1}),\ \Psi(z, \mathbf{1})\right\}, \quad z \in \mathcal{Z}, \qquad \text{(III.5.8)}$$

involving the $M_1 M_2 \times M_1 M_2$ matrix $\tilde{P} \equiv \{\tilde{p}_{zy},\ z, y \in \mathcal{Z}\}$, where $z \in \tilde{\mathbb{C}}^* := \mathcal{Z} \setminus \tilde{\mathbb{S}}^*$ if, and only if, the maximum is attained by the first term in braces. The definition (III.5.7) is of course made with the intention that $\tilde{v}(\cdot, \mathbf{1})$ approximate the true value function $v(\cdot, \mathbf{1})$ as $M_k \to \infty$, $h_k \to 0$, $k = 1, 2$, and $\Delta \to 0$. See Kushner and Dupuis (2001) for details of several types of conditions under which this convergence is guaranteed in more general control problems, but the gist of the matter is as follows.

Given an interpolation interval Δ, the one-step transition mechanism of $\tilde{X}$ is chosen to make $\tilde{X}$ and X consistent in the sense that the first and second moments of the one-step increment of $\tilde{X}$ agree with those of the increment over $[0, \Delta]$ of X, at least to within a term that becomes arbitrarily small for small Δ. One then proves a weak convergence result whereby the chain $\tilde{X}$, under controls adapted to $\tilde{X}$, converges weakly to the controlled process X. By this, cost functionals depending on $\tilde{X}$, such as that appearing in $[\cdot]$ in (III.5.7), can be shown to converge to a functional of X.

The problem (III.5.7) is a particularly simple Markov Chain control problem where the control $\tilde{\Phi}(0)$ at time 0 is to either take $\tilde{\Phi}(0) = \mathbf{1}$ and allow the chain one more transition, after which the problem restarts with the chain in a different state, or else to take at least one component of $\tilde{\Phi}(0)$ to be zero, whereby $\tilde{\tau} = 0$ and the problem becomes one-dimensional with explicitly-known value $\Psi(x, \mathbf{1})$.

Such an optimal stopping problem is an example of a more general class of Markov decision problems, which are described briefly below for the case of a finite state space. For a more comprehensive discussion, see Bertzekas (1976; Ch. 6), and also Cottle, Pang and Stone (1992) for the connection with variational inequalities.

Suppose S is a finite set, and to each $x \in S$ associate a finite set $\mathcal{U}(x)$. The elements of $\mathcal{U}(x)$ are *controls* admissible at x, so that each pair $(x, u) \in S \times \mathcal{U}(x)$ can be associated with a probability distribution $p_u(x, \cdot)$ on S. Now suppose we start with $X_0 = x \in S$ and set up an *admissible* policy $\pi = (\mu_0, \mu_1, \ldots)$, where for each $k \geq 0$, μ_k is a function on S satisfying

$$\mu_k : S \to \mathcal{U} := \cup_{x \in S} \mathcal{U}(x); \quad \mu_k(x) \in \mathcal{U}(x), \quad x \in S.$$

Construct $X \equiv (X_k)_{k \geq 0}$ by employing the transition mechanism

$$\mathbb{P}\Big(X_{k+1} = \cdot \;\Big|\; X_k = x\Big) := p_{\mu_k(x)}(x, \cdot); \tag{III.5.9}$$

this makes X a (generally time-inhomogeneous) Markov Chain with state space S.

Given the initial state x, and the admissible policy π, define an objective function $J_\pi : S \to \mathbb{R}$ by

$$J_\pi(x) := \mathbb{E}\Big[\sum_{k=0}^{\infty} \theta^k R(\, X_k, \mu_k(X_k)\,) \;\Big|X_0 = x\Big], \tag{III.5.10}$$

where $R : S \times \mathcal{U} \to \mathbb{R}$ is a reward function (defined on a finite set and therefore bounded), $\theta \in (0,1)$ is a discount factor, and expectation is in the law characterized by (III.5.9).

The goal is to choose the policy π so as to attain

$$J^*(x) := \max_\pi J_\pi(x), \qquad x \in S, \tag{III.5.11}$$

and the computational methods we describe now can be used to obtain the value function J^* in the case when the maximizing admissible policy π^* in (III.5.11) is *stationary*, that is, of the form $\pi = (\mu, \mu, \ldots)$ with $\mu(x) \in \mathcal{U}(x)$ for each $x \in S$. To this end, define the operators T, T_μ, such that for each function $J : S \to R$ and each $\mu : S \to \mathcal{U}$ with $\mu(x) \in \mathcal{U}(x)$, $\forall x \in S$, we have

$$\begin{aligned} T(J)(x) &:= \max_{u \in \mathcal{U}(x)} \Big\{R(x,u) + \theta \sum_{y \in S} p_u(x,y) J(y)\Big\} \\ T_\mu(J)(x) &:= \Big\{R(x,\mu(x)) + \theta \sum_{y \in S} p_{\mu(x)}(x,y) J(y)\Big\}. \end{aligned} \tag{III.5.12}$$

The dynamic programming equation for(III.5.11) now tells us that the value function J^* is a fixed point of the operator T:

$$T(J^*) = J^*, \tag{III.5.13}$$

and the value iteration method is based on solving this equation by successive approximations.

Value Iteration Algorithm

Start with an arbitrary function $J : S \to \mathbb{R}$, and form the sequence

$$T(J), T^2(J), \ldots, T^k(J), \ldots.$$

This sequence converges pointwise on S to the solution J^* of (III.5.13). Upper and lower bounds for J^* in terms of the successive iterates $T^k(J)$ exist and can be used to check the progress of the iteration, which is stopped when the computed bounds are sufficiently tight.

Policy Improvement Algorithm

Policy Improvement, on the other hand, starts with a function $\mu^0 : S \to \mathcal{U}$ characterizing a stationary admissible policy $\pi^0 = (\mu^0, \mu^0, \ldots)$, and produces a sequence $\mu^1, \mu^2, \ldots$ of functions for which the corresponding sequence of objective values *increases strictly*, whenever such an increase is possible. The procedure is as follows.

Given a stationary policy $\pi = (\mu, \mu, \ldots)$, write $J_\mu \equiv J_\pi$ for the corresponding objective in (III.5.10), and define $\bar{\mu}$ by setting

$$\begin{aligned} &R(x, \bar{\mu}(x)) + \theta \sum_{y \in S} p_{\bar{\mu}(x)}(x, y) J_\mu(y) \\ &= \max_{u \in \mathcal{U}(x)} \Big\{ R(x, u) + \theta \sum_{y \in S} p_u(x, y) J_\mu(y) \Big\}, \quad x \in S. \end{aligned} \tag{III.5.14}$$

If $J_{\bar{\mu}}$ is the value of the objective corresponding to $\bar{\mu}$, then we have

$$J_{\bar{\mu}} \geq J_\mu.$$

If π is not an optimal stationary policy, then $(\bar{\mu}, \bar{\mu}, \ldots)$ is a policy that is strictly better than π, and the procedure can be repeated. If μ is optimal, then $\bar{\mu} = \mu$ and the algorithm stops after computing $J_{\bar{\mu}} = J^*$.

In actual computations, both the sets S and $\mathcal{U}$ will obviously be finite, and the policy improvement algorithm will in this case produce J^* in a finite number of steps. Also, the operators T and T_μ defined above can be expressed in terms of matrix operations, so computing the objective function J_μ corresponding to a stationary policy μ involves solving a linear system

$$T_\mu(J_\mu) = J_\mu.$$

The ease with which this can be done in practice depends very much on the structure of the transition matrices $P_\mu \equiv \{p_{\mu(x)}(x,y),\ x,y \in S, \mu(x) \in \mathcal{U}(x)\}$ Generally, the required inversion can be done efficiently and quickly if the matrices involved are sparse or of small dimensions. If not, it may be more effective to use value iteration to obtain an approximate solution to the value function.

For our particular bi-variate problem, with state-space $\mathcal{Z}$, the set of controls $\mathcal{U}$ consists simply of the four possible values for $\tilde{\Phi}$, whereas the admissible controls at the boundary $\partial\mathcal{Z}$ of $\mathcal{Z}$ depend on the artificial conditions that one imposes. The examples we consider have values known in closed form when there is only one firm present, so we shall only need to describe the transition mechanism $\tilde{P} \equiv \tilde{P}_{\mathbf{1}}$ conditional on the control $\tilde{\Phi} = \mathbf{1}$.

We describe below the different transition mechanisms satisying the *local consistency conditions* required to make $\tilde{X}$ approximate continuous-time processes of interest. Throughout, we assume that the chain $\tilde{X}$ is *absorbed* at the boundary $\partial\mathcal{Z}$, where we impose a Dirichlet-type condition. The discount factor in (III.5.8) is taken to be

$$\tilde{\beta} := \exp\left(-\delta\ \Delta\right).$$

III.5.4. Special case I: Brownian Motion

Suppose the process X satisfies

$$X_k(t) = \mu_k + \sigma_k W_k(t), \quad k = 1, 2,$$

where $\mu_k \in \mathbb{R}$, $\sigma_k \in \mathbb{R}^+ \setminus \{0\}$, $k = 1, 2$, $\mathbb{E}[W_1(t)W_2(t)] = \rho\sigma_1\sigma_2 t$, $\rho \in [-1, 1]$. A Markov chain $\tilde{X}$ approximating X is obtained as follows. Choose $\lambda \geq 1$, $\Delta > 0$, and pick $h = (h_1, h_2)^T$ and such that $\frac{h_1}{\sigma_1} = \frac{h_2}{\sigma_2} = \lambda\sqrt{\Delta}$. Now define

$P^{BM} \equiv \{(p^{BM}_{(i,j)(k,m)}),\ i,k \in \{0,\dots,M_1-1\},\ j,m \in \{0,\dots,M_2-1\}\}$ by

$$p^{BM}_{(i,j)(k,m)} :=$$
$$\begin{cases} \frac{1}{4\lambda^2}[(1+\rho)+(\frac{\mu_1}{\sigma_1}+\frac{\mu_2}{\sigma_2})\lambda\sqrt{\Delta}] & \text{if } k=i+1, m=j+1,\ (x_i,y_j)\notin \partial\mathcal{Z} \\ \frac{1}{4\lambda^2}[(1-\rho)+(\frac{\mu_1}{\sigma_1}-\frac{\mu_2}{\sigma_2})\lambda\sqrt{\Delta}] & \text{if } k=i+1, m=j-1,\ (x_i,y_j)\notin \partial\mathcal{Z} \\ \frac{1}{4\lambda^2}[(1+\rho)+(-\frac{\mu_1}{\sigma_1}-\frac{\mu_2}{\sigma_2})\lambda\sqrt{\Delta}] & \text{if } k=i-1, m=j-1,\ (x_i,y_j)\notin \partial\mathcal{Z} \\ \frac{1}{4\lambda^2}[(1-\rho)+(-\frac{\mu_1}{\sigma_1}+\frac{\mu_2}{\sigma_2})\lambda\sqrt{\Delta}] & \text{if } k=i-1, m=j+1,\ (x_i,y_j)\notin \partial\mathcal{Z} \\ 1-\lambda^{-2} & \text{if } (x_i,y_j)=(x_k,y_m)\notin \partial\mathcal{Z} \\ 1 & \text{if } (x_i,y_j)=(x_k,y_m)\in \partial\mathcal{Z} \\ 0 & \text{otherwise.} \end{cases} \tag{III.5.15}$$

It is easily checked that the increment $\tilde{X}_n - \tilde{X}_{(n-1)}$ has first and second moments equal to those of $X_{n\Delta} - X_{(n-1)\Delta}$, up to a term of order $O(\Delta^2)$, and that all probabilities lie in $[0,1]$ if Δ is chosen small enough. Notice that $\tilde{X}$ is absorbed in $\partial\mathcal{Z}$. We take our boundary condition to be

$$\tilde{v}(z,\mathbf{1}) = \Psi(z,\mathbf{1}) \vee v^{\sharp}(z,\mathbf{1}), \quad z \in \partial\mathcal{Z}, \tag{III.5.16}$$

that is, the value on $\partial\mathcal{Z}$ is the larger of the one-firm value and the value of the linear strategy outlined in Proposition III.5.1.

The above discretization is often referred to as the five-point formula, and can be found in Kwok (1998). Nelson and Ramaswamy (1990) show how a similar discretization can be used for general diffusions while retaining the simplicity afforded by a grid with regular spacing.

In practice, the transition matrix represented by (III.5.15) is very sparse, and this makes it feasible to use the policy improvement algorithm to solve the dynamic programming equation (III.5.8) for $\tilde{X}$.

III.5.5. Special case II: CPP with independent negative exponential jumps

Consider now the case when for $k = 1, 2$,

$$X_k(t) = \mu_k t + Z_k(t)$$

with $\mu_k > 0$ and Z_k a CPP of negative exponential jumps of mean $-c_k^{-1}$ arriving at rate a_k, as discussed in Section III.4.4. For now, we assume the jump processes Z_1 and Z_2 are independent.

Given $\Delta > 0$, choose $h_k = \mu_k \Delta$, $k = 1, 2$. We approximate the jump distributions of X by geometric random variables absorbed in $\partial\mathcal{Z}$. To this end, let $(1 - q_k) \equiv p_k := \exp(-c_k h_k)$, $k = 1, 2$, and define $P^{CPPI} \equiv \{(p^{CPPI}_{(i,j)(k,m)}),\ i, k \in \{0, \ldots, M_1 - 1\},\ j, m \in \{0, \ldots, M_2 - 1\}\}$ by

$$p^{CPPI}_{(i,j)(k,m)} := \begin{cases} [1 - (a_1 + a_2)\Delta] & \text{if } k = i+1,\ m = j+1,\ (x_i, y_j) \notin \partial\mathcal{Z} \\ a_1\Delta\ q_1 p_1^{i-k} & \text{if } 1 \le k \le i,\ m = j+1,\ (x_i, y_j) \notin \partial\mathcal{Z} \\ a_1\Delta\ p_1^{i} & \text{if } k = 0,\ m = j+1,\ (x_i, y_j) \notin \partial\mathcal{Z} \\ a_2\Delta\ q_2 p_2^{j-m} & \text{if } k = i+1,\ 1 \le m \le j,\ (x_i, y_j) \notin \partial\mathcal{Z} \\ a_2\Delta\ p_2^{j} & \text{if } k = i+1,\ m = 0,\ (x_i, y_j) \notin \partial\mathcal{Z} \\ 1 & \text{if } (x_i, y_j) = (x_k, y_m) \in \partial\mathcal{Z} \\ 0 & \text{otherwise.} \end{cases} \tag{III.5.17}$$

Value improvement has to be used to solve (III.5.8) with the matrix P^{CPPI}. We take the boundary condition

$$\tilde{v}(z, \mathbf{1}) = \Psi(z, \mathbf{1}) \vee v^\sharp(z, \mathbf{1}), \quad z \in \partial\mathcal{Z}. \tag{III.5.18}$$

Here, $v^\sharp(z, \mathbf{1})$ is not known in closed form unless $a_1 = a_2$, $c_1 = c_2$, so we compute it as if the jump components of X *were* identical with jump arrival rate $a = (a_1 + a_2)/2$ and mean size $-c^{-1} = -(c_1^{-1} + c_2^{-1})/2$.

III.5.6. Special case III: CPP with correlated negative exponential jumps

As a generalisation of the previous case, we consider a CPP where jumps follow the Gumbel bi-variate exponential distribution described in Balakrishnan, Johnson and Kotz (2000). We assume that jumps of X occur at rate a; if τ is a jump time for X, then $\Delta X := X_\tau - X_{\tau-}$ is a bivariate distribution such that $(\Delta X)_k$ is exponential of mean c_k^{-1}, and with conditional density

$$f_{\Delta X_2|\Delta X_1}(x_2|x_1) = c_2 e^{-c_2 x_2(1+\theta c_1 x_1)}\{(1+\theta c_1 x_1)(1+\theta c_2 x_2) - \theta\}$$

where $\theta \in [0,1]$ is a parameter controlling the dependence.

We approximate the exponential marginal distribution of X_1 in the same way as above and the conditional density by a piecewise constant function such that integrals agree on each sub-partition of $[0,(M_2-1)h_2]$ implied by h_2. Thus, set $P^{CPPD} \equiv \{(p^{CPPD}_{(i,j)(k,m)}),\ i,k \in \{0,\dots,M_1-1\},\ j,m \in \{0,\dots,M_2-1\}\}$ where

$$p^{CPPD}_{(i,j)(k,m)} := \begin{cases} [1-a\Delta] & \text{if } k=i+1,\ m=j+1,\ (x_i,y_j) \notin \partial\mathcal{Z} \\ a\Delta q_1 p_1^{i-k}\ p_2(j,m|i+1-k) & \text{if } 1 \le k \le i,\ (x_i,y_j) \notin \partial\mathcal{Z} \\ a\Delta p_1^{i}\ p_2(j,m|i+1) & \text{if } k=0,\ (x_i,y_j) \notin \partial\mathcal{Z} \\ 1 & \text{if } (x_i,y_j) = (x_k,y_m) \in \partial\mathcal{Z} \\ 0 & \text{otherwise,} \end{cases} \tag{III.5.19}$$

where for each $t = 1,\dots,i+1$, we set $\tilde{p}_t := \exp(-c_2 h_2(1+\theta t c_1 h_1))$ and

$$p_2(j,m|t) := \begin{cases} (\tilde{p}_t)^{n-1}[(\theta c_2 n h_2 + 1)(1-\tilde{p}_t) - \theta c_2 h_2] & \text{if } 1 \le n := j+1-m \le j,\ m>0 \\ 1 - \sum_{n=1}^{j} \tilde{p}_2(j,n|t) & \text{if } m=0. \end{cases} \tag{III.5.20}$$

The boundary condition we use is the same as in (III.5.18).

III.6. Numerical Results

We now present numerical results obtained from solving the discrete-time two-firm model, as described in Section III.3 as well as the two-firm continuous-time model of section III.5.

III.6.1. Results for IID output

Recall the key simplifying assumption made in Section III.3 that firm output is an IID sequence of normally distributed random vectors. We simplify notation and denote by $X = (X_1, X_2)^T \sim N(\mu, V)$ the output at any given time n, $n \geq 0$, where

$$\mu = (\mu_1, \mu_2)^T \in \mathbb{R}^2, \quad V = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix},$$

with $\sigma_1, \sigma_2 > 0$, the correlation $\rho \in (-1, 1)$, and V positive definite.

From Section III.3, the value to be computed for each $x \in \mathbb{R}^2$ is given by

$$v(x, \mathbf{1}) = \max\Big\{U(x \cdot \mathbf{1}) + \beta K,\ \Psi(x, \mathbf{1})\Big\} \tag{III.6.1}$$

where $K = K_{\{1\}} = \mathbb{E}v(X, \mathbf{1})$ and

$$\Psi(x, \mathbf{1}) = v(I_1 \cdot x, I_1) \vee v(I_2 \cdot x, I_2),$$

with $I_k = I_{\{k\}}$, $k = 1, 2$. The function $\Psi(\cdot)$ is available in closed form from Section III.3.1 once the exercise boundaries $b^* = (b_1^*, b_2^*)^T$ are obtained from (III.3.4).

For a large number of simulated model parameters μ, V, Γ and β, we estimated numerically the constant K in (III.6.1) using the procedure described in Section III.3.2. For the iteration procedure expectations were estimated using Monte Carlo (MC) simulation with antithetic variables. For some examples we complemented this by numerical integration on a fine grid.

To assess whether the contagion region is empty (see (III.3.7)), we computed

$$V_{\mathbb{D}^*} := \frac{[v(b^*, \mathbf{1}) - U(0)/(1-\beta)]}{|U(0)/(1-\beta)|}, \tag{III.6.2}$$

this being the relative value at b^* in excess of the value of default at b^*. We also report the following probabilities of interest:

1. $p_i := \mathbb{P}[v(X, \mathbf{1}) = \Psi(X, \mathbf{1}) = U(X \cdot I_k) + \beta \mathbb{E} v(X \cdot I_k, I_k),\ k = 3 - i],\ i = 1, 2$
2. $p_{12} := \mathbb{P}[v(X, \mathbf{1}) = \Psi(X, \mathbf{1}) = U(0)/(1 - \beta)]$
3. $q := 1 - p_1 - p_2 - p_{12} = \mathbb{P}[v(X, \mathbf{1}) = U(X, \mathbf{1}) + \beta K]$
4. $p_{\mathbb{D}^*} := \mathbb{P}[X \in \mathbb{D}^*]$
5. $d_k := \mathbb{P}[X_k \leq b_k^*],\ k = 1, 2.$

Here, p_k is the one-step probability in the two-firm model that the k'th firm *only* is defaulted. Both firms are defaulted simultaneously on a set of probability p_{12}, whereas with probability q no default happens. For comparison, we report also the probabilities d_k, $k = 1, 2$, where d_k is the probability of default in a one-firm model for firm k.

Table A.1 shows results of some instances of the model for which $\mu > 0$ [21]. MC estimates are compared to ones from numerical integration (NI). The MC estimates for the constant K are different by not more than 1% from their NI values. MC estimates for other quantities are less reliable, because the numbers involved are so tiny; nevertheless in only a few instances do the MC and NI procedures not agree on the presence of a non-zero value for $V_{\mathbb{D}^*}$, which is what we are after here.

It is apparent that non-zero values for $V_{\mathbb{D}^*}$ and $p_{\mathbb{D}^*}$, indicating a contagious effect, occur for large negative values of the correlation ρ. Also, as expected, the probability $p_k + p_{12}$ that the k'th firm is defaulted in the two-firm model is significantly lower than the probability of default d_k that would obtain in a one-firm model. Table A.2 presents further simulations, but omitting the NI calculations. With $\mu < 0$, simulations resulted in no parameter values for which $V_{\mathbb{D}^*} > 0$. In Table A.3 we present some simulations for which μ has components of opposite sign. Again the values $V_{\mathbb{D}^*}$ are larger when $\mu \cdot \mathbf{1}$ is positive. It seems from these numbers that what drives contagion in this two-firm model is the presence of negatively-correlated assets bearing positive total average output[22].

Figures III.2 and III.3 show plots of the value surface $v(\cdot, \mathbf{1})$, as well as the corresponding contours, for two particular instances of model parameters chosen from Table A.2. There is a contagious effect for these parameters, as is evident

[21]... inequality to be understood component-wise ...

[22]Although a few instances arise in Table A.3 where correlation is positive, the value $V_{\mathbb{D}^*}$ that results is too small to be conclusive. Moreover, the probability $p_{\mathbb{D}^*}$ is 0 for these instances.

from the form of the lowest level of the contour plot. The linearity of the contours suggest that inside the continuation region the value function depends on X only through a linear combination of its components. In fact, the location of the exercise boundary delineating the region $\mathbb{D}^*$ is consistent with the critical level expected if only a single firm were available, with output $X \cdot \mathbf{1}$.

III.6.2. Results for Brownian output

Value Iteration and / or Policy Improvement were used to obtain numerically the value function (III.5.2) for the two-firm model with Brownian output. The discretization for Brownian dynamics was done as explained in Section III.5.4.

Figure III.4 shows the form of the value function for a pair of firms with identical Brownian dynamics but with negative correlation. The contagion region is empty in this case, as confirmed by the first panel of the contour plot, Figure III.5. To benchmark the numerical procedure, we also solved numerically for the optimal linear strategy of Section III.5.2 (with the free parameter m of that Section equal to 1). The relative difference between the numerically estimated value function for this suboptimal strategy and its known analytical form is at most 4.59×10^{-4}; the location of the exercise boundary coincides with what we expect $b^\sharp$ to be, as can be confirmed from the second panel of Figure III.5 by reading off the sum of the values on the axes along the lowest contour line.

All other model instances we tried resulted in an empty contagion region and a value function of a form similar to what is seen in Figure III.4. Using the discretization scheme described in Nelson and Ramaswamy (1990) we even computed a numerical solution allowing the components of the process X to be correlated Ornstein-Uhlenbeck processes, but the contagion regions seen for this type of models were empty also.

III.6.3. Results for Lévy output

Using the discretization schemes described in Sections III.5.5 and III.5.6, we computed numerically the value function (III.5.2) when the underlying process X is a CPP with (possibly correlated) negative jumps having a generalized exponential distribution.

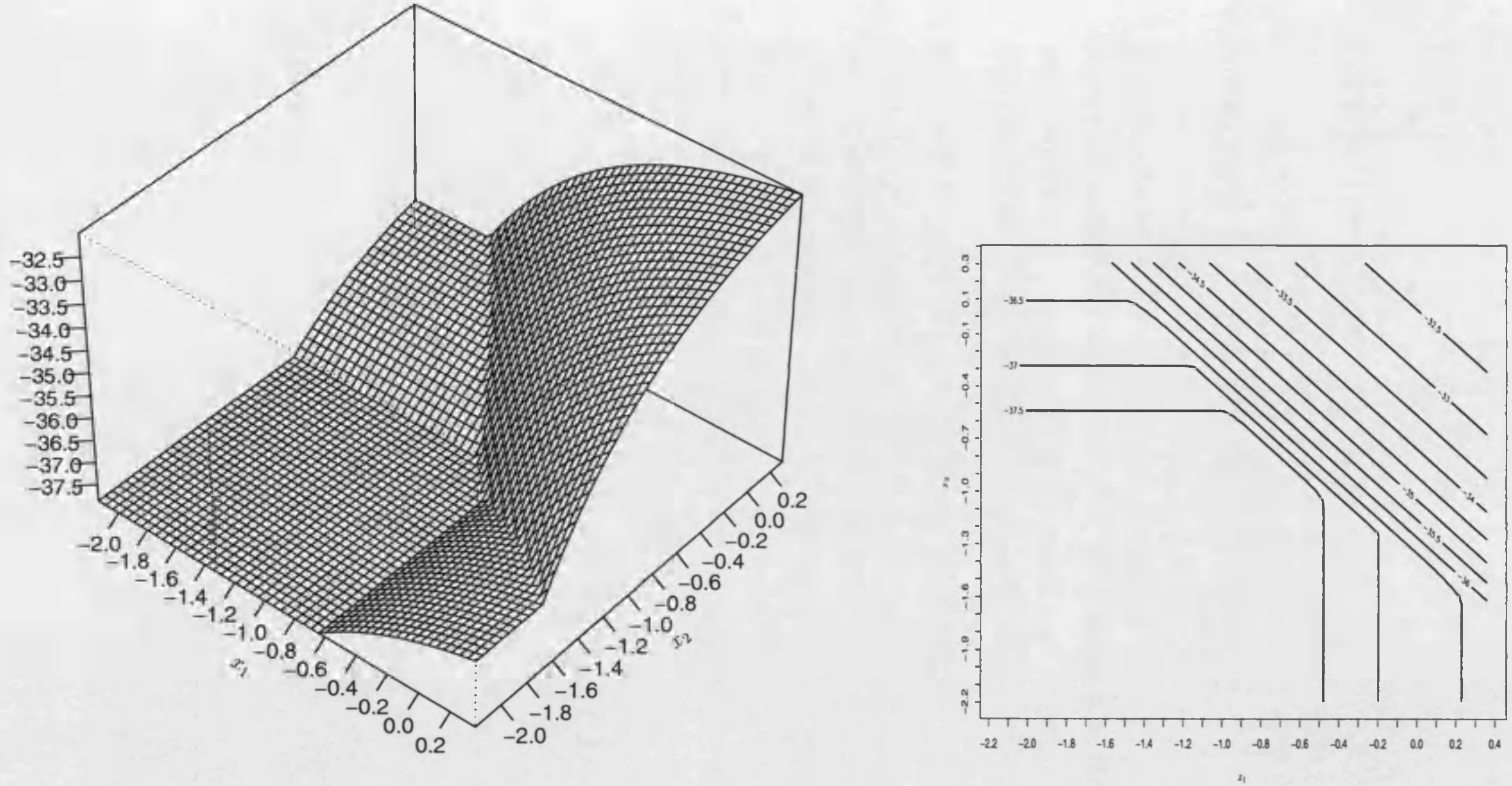


Figure III.2: Value function surface (left panel) and level surface plot (right panel) for IID model. Parameters are $\sigma_1 = 0.696$, $\sigma_2 = 0.645$, $\mu_1 = 0.192$, $\mu_2 = 0.197$, $\rho = -0.510$, $\beta = 0.974$, $\Gamma = 1.284$. The constant K is estimated at -32.403, with $V_{\mathbb{D}^*} = 0.039$, $p_{\mathbb{D}^*} = 1.36 \times 10^{-4}$. Numerical integration was used on a grid of 400×400 nodes, with spacing 0.03 in each coordinate direction.

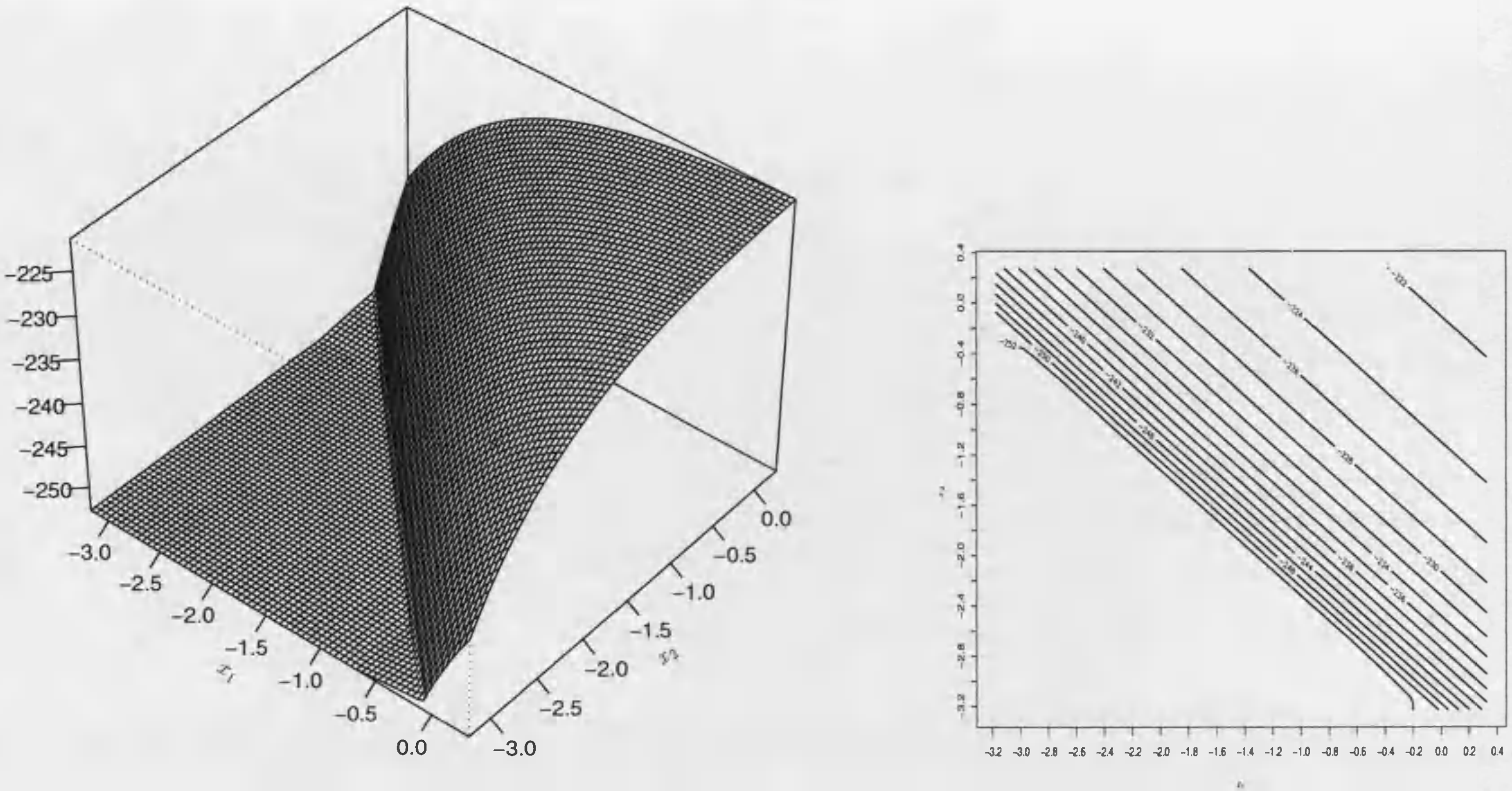


Figure III.3: Value function surface (left panel) and level surface plot (right panel) for IID model. Parameters are $\sigma_1 = 0.710$, $\sigma_2 = 0.510$, $\mu_1 = 0.146$, $\mu_2 = 0.112$, $\rho = -0.705$, $\beta = 0.996$, $\Gamma = 1.038$. The constant K is estimated at -221.763, with $V_{\mathbb{D}^*} = 0.109$, $p_{\mathbb{D}^*} = 8.0 \times 10^{-6}$. Numerical integration was used on a grid of 400×400 nodes, with spacing 0.03 in each coordinate direction.

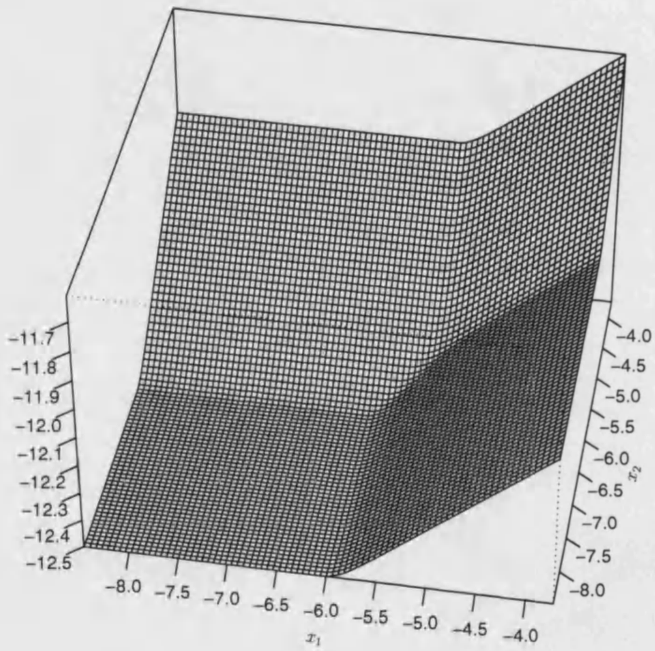
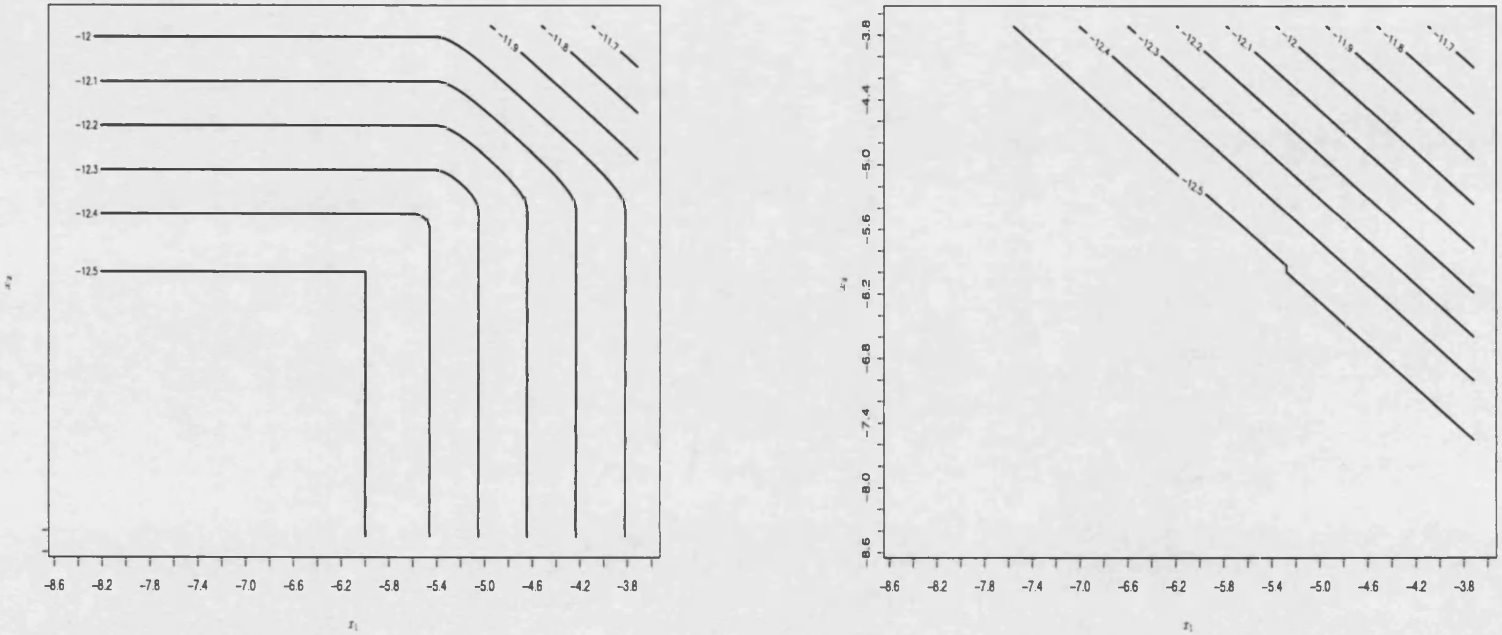


Figure III.4: A plot of the value function for firms having Brownian outputs with identical characteristics $\sigma_1 = \sigma_2 = 0.4$, $\mu_1 = \mu_2 = 0.5$, and a negative correlation $\rho = -0.5$. Remaining parameters are $\delta = 0.08$, $\Gamma = 0.02$. The solution was computed on a 200×200 grid, the central 80×80 section of which is shown here. Grid spacing is $h_1 = h_2 = 0.06$, consistent with timestep $\Delta = 0.01$. The one-firm exercise boundaries are $b_1^* = b_2^* = -6.03$ (analytic) and $b_1^* = b_2^* = -6.00$ (numeric). A Value Iteration scheme was used.



(a) Level surface plot, value function

(b) Level surface plot, linear strategy

Figure III.5: Level surface plots for the value function of Figure III.4 as well as for the sub-optimal linear strategy. The analytical exercise boundary for the latter is $b^{\sharp} = -11.22$. A Value Iteration scheme was used.

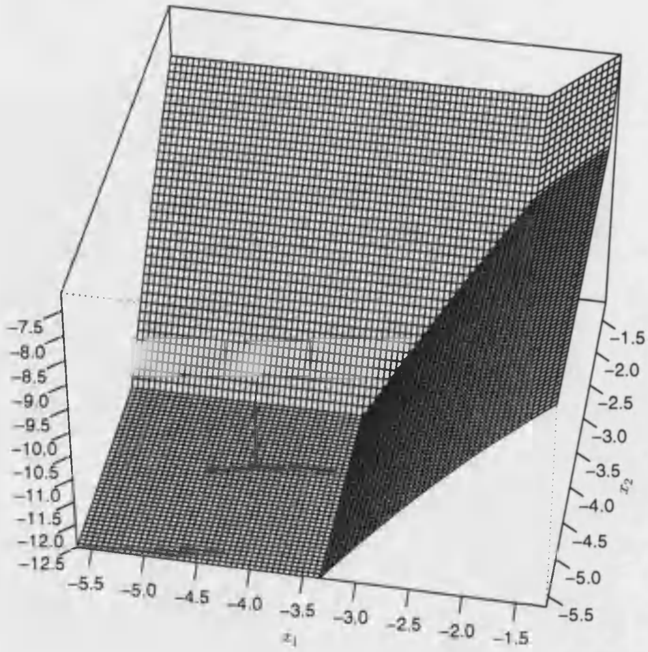
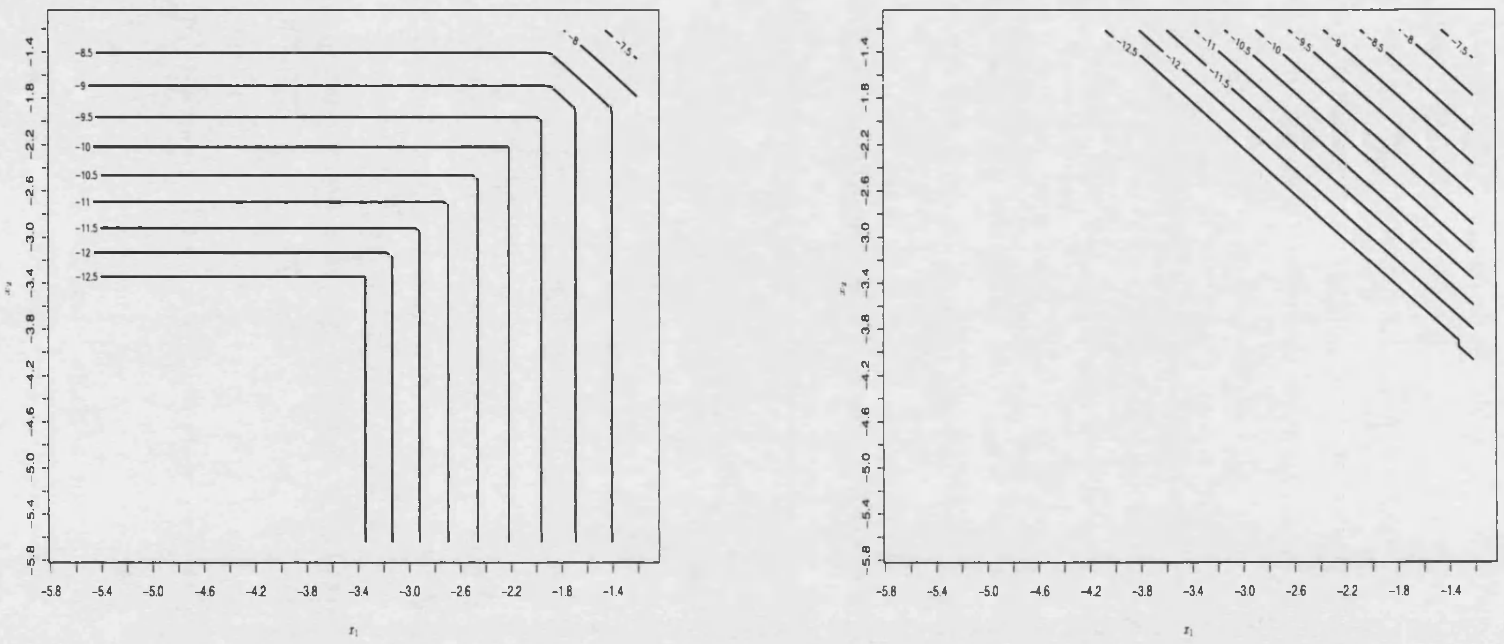


Figure III.6: A plot of the value function for firms whose outputs are a pure drift added to independent CPP's with negative exponential jumps. Parameters are $a_1 = a_2 = 0.12$, $\mu_1 = \mu_2 = 0.4$, $c_1 = c_2 = 6$, $\delta = 0.08$, $\Gamma = 0.2$. The solution was computed on a 181×181 grid, the central 80×80 section of which is shown here. Grid spacing is $h_1 = h_2 = 0.056$, timestep $\Delta = 0.14$. The exact one-firm exercise boundaries are $b_1^* = b_2^* = -3.34$. A Value Iteration scheme was used.

We present in Figure III.6 the value function resulting for a pair of firms with identical and independent dynamics, whose parameters are as captioned. The left panel of Figure III.7 gives the level surface plot for the value function; the right shows the level surfaces for the (suboptimal) linear strategy, the value of which is known in closed form given our choice of independent, identically distributed jumps for the components of X. As can be confirmed from the position of the lowest contour in the right panel of Figure III.7, the exercise boundary is to within one grid-square from where we expect it to be analytically, giving rise to a relative error of at most 2% between the analytic and numerical values for the suboptimal linear rule.

Again, we found no instances of the model with Lévy dynamics admitting a non-empty contagion region, even when we allowed the process to have jumps that are correlated, as detailed in Section III.5.6.



(a) Level surface plot, value function

(b) Level surface plot, linear strategy

Figure III.7: Level surface plots for the value function of Figure III.6 as well as for the sub-optimal linear strategy. The analytical exercise boundary for the latter is $b^\sharp = -5.33$. A Value Iteration scheme was used.

III.7. Concluding Remarks

We have investigated the effect of a limited liability assumption on an equilibrium where agents may rationally choose to default firms providing random unbounded cashflows with a view to maximizing an expected utility functional. The form of the solution is that the cashflows have value while they remain larger than endogenously-determined critical levels. Once these are breached, one *or possibly more* of the firms are defaulted, at which point their output is replaced by cashflows that are identically zero.

Because of our assumption of an exponential form for the utility, the equilibrium can in principle be computed explicitly in a market with a *single* output stream that can be as general as a spectrally negative Lévy Process. Also because of our choice of utility, the solution to the individual agent's optimization problem extends to a multi-agent market[23] where equilibrium entails in particular that all agents agree on when default is enforced. The equilibrium price for the cashflow in this market encompasses the net present value of output *and* the value of the option that the agents have to default. As expected, the value of the default option is positive, and decreases with levels of output.

While analysis takes us a long way in the one-firm model, the only possible attempt at solution when even just two firms are present is numerical. As in the one-firm case, the two-firm solution is characterized by an exercise boundary which determines levels of output at which rationally-behaving agents are indifferent between defaulting at least one firm or not. The key point is that in this model, such an exercise boundary arises endogenously from preferences of the agents and the output dynamics. In turn, this introduces endogenous dependence between the default status of the two firms, an effect that is interesting in its own right. Moreover, there is nothing to exclude to possibility that the kind of dependence arising can also cause contagious effects whereby default of one firm immediately induces that of another. Said another way, it might happen that the default option in a two-firm market is worth more than its intrinsic value for the same levels of output at which the option would be exercised in a one-firm market for either firm.

As can be seen from the plots in Section III.6, the exercise boundary in the two-firm model is generally a distorted wedge-shape; this of course confirms de-

[23] see Appendix A.4

pendence between defaults does arise. Unfortunately, the only instances where we saw *contagious* effects were of discrete-time models in which the output constituted an IID sequence of normal random variables, and from the numerical simulations we have described, these effects seem to be significant only for negative correlation between outputs from different firms. In continuous time, not a single instance with contagious effects arose for the kinds of dynamics (BM, CPP) that we tried. A condition sufficient for contagion given in Proposition III.4.15 turned out also not to hold for several non-exponential utilities U and Brownian dynamics. Because in the discrete-time IID case it is the cumulative output, rather than the output process itself, that has independent and stationary increments, we computed also the value function for mean-reverting (non-Lévy) OU dynamics in continuous time, but still this produced none of the contagious effects seen in the IID model.

The basic process X, which we have used to represent net output from firms, can be interpreted as dividend payments less coupons for leveraged firms. The model we have presented assumes that the representative agent solves his stopping problem by defaulting when the expected utility of X equals that of an output rate identically equal to 0. Intuitively, in a one-firm model this means that the agent exits when there is no value left in the firm. If instead the agent were assumed to exit when a small but *positive* amount of value remained, the mathematical structure of the solution would remain unchanged. In effect this would entail that a defaulted firm provide a constant positive output rate (representing what is paid to holders of defaulted bonds) as opposed to no output at all, and the result would be to shift upwards the critical levels b^* we have computed.

Appendix

A.1. Some results for the CIR diffusion.

We prove here the result, referred to in Section II.4, that under some mild conditions on the model parameters, the change-of-measure process induced by the state-price density ζ is a true martingale.

Note first that the process Z appearing in (II.4.4) is a non-negative local martingale. Define the stopping times

$$\tau_n := \inf\{t : \frac{R\sigma}{\sqrt{\Delta}} > n\}. \tag{A.1.1}$$

Then clearly, for each $n > 0$, the stopped process $Z^n \equiv (Z_{t\wedge\tau_n})_{t\geq 0}$ is a true martingale and can be used to define a probability $\tilde{\mathbb{P}}^n$ equivalent to $\mathbb{P}$ on every $\mathcal{F}_T$, $T > 0$. *If* for every $T > 0$ we have the condition

$$\tilde{\mathbb{P}}^n[\tau_n < T] \to 0 \text{ as } n \to \infty, \tag{A.1.2}$$

then we can conclude that Z is a true martingale; see Hobson and Rogers (1998). For our specific process Z, the condition displayed above will hold if the process Δ never reaches 0 under the measure induced by Z.

We have now the following result.

Lemma A.1.1. *The condition*

$$\frac{2A}{\sigma^2} \geq 2R + 1 \tag{A.1.3}$$

is necessary and sufficient for the local martingale Z defined at (II.4.4) to be a martingale.

Proof. Firstly, suppose that the local martingale Z is actually a martingale. The effect of the change of measure is to add a drift to dB:

$$dB = d\tilde{B} - \frac{R\sigma}{\sqrt{\Delta}}\, dt$$

where $\tilde{B}$ is a $\tilde{\mathbb{P}}$-Brownian motion, so that Δ solves the SDE

$$d\Delta = \sigma\sqrt{\Delta}d\tilde{B} + (A - R\sigma^2 - \beta\Delta)dt \tag{A.1.4}$$

in the probability $\tilde{\mathbb{P}}$. This SDE for Δ is exact, and is of the same general (CIR) form as the original SDE. Because of the standing assumption (II.2.3), we have for any $T > 0$ that

$$\mathbb{P}[\Delta_t > 0 \text{ for all } 0 \leq t \leq T] = 1,$$

and since $\tilde{\mathbb{P}}$ is equivalent to $\mathbb{P}$ on any $\mathcal{F}_T$, we have to have

$$\tilde{\mathbb{P}}[\Delta_t > 0 \text{ for all } 0 \leq t \leq T] = 1.$$

The necessary and sufficient condition for this is (A.1.3), because in $\tilde{\mathbb{P}}$, Δ is a time-change of a squared Bessel process of dimension $4(A - R\sigma^2)/\sigma^2$.

For the converse, suppose that condition (A.1.3) is satisfied. Consider the stopping times τ_n in (A.1.1). For each positive integer n, the process Z^n is a martingale and induces a measure $\tilde{\mathbb{P}}^n$. In this measure, Δ satisfies SDE

$$d\Delta = \sigma\sqrt{\Delta}d\tilde{B} + \big((A - R\sigma^2 - \beta\Delta)I_{\{t<\tau_n\}} + (A - \beta\Delta)I_{\{t>\tau_n\}}\big)dt. \tag{A.1.5}$$

The condition (A.1.3) ensures that (A.1.2) holds; the Hobson and Rogers (1998) result alluded to above allows us now to conclude that Z is a true martingale. ■

Lemma A.1.2. *The solution to (II.2.2)*

$$d\Delta_t = \sigma\sqrt{\Delta_t}dB_t + (A - \beta\Delta_t)dt$$

is an ergodic diffusion on $(0, \infty)$ *with invariant law* $\Gamma(2A/\sigma^2, 2\beta/\sigma^2)$. *The expectation*

$$\mathbb{E}^x \int_0^\infty e^{-\rho t}\Delta_t^\theta\, dt \tag{A.1.6}$$

is finite for all $x > 0$ if and only if

$$\theta + \frac{2A}{\sigma^2} > 0 \tag{A.1.7}$$

Proof. The invariant density π of Δ solves the adjoint equation

$$\mathcal{G}^*\pi \equiv D^2\left[\frac{1}{2}\sigma^2 x\pi(x)\right] - D\left[(A - \beta x)\pi(x)\right] = 0$$

and it is a simple exercise to solve this for π to obtain a density

$$\pi(x) = x^{-1+2A/\sigma^2} e^{-2\beta x/\sigma^2} / \Gamma(2A/\sigma^2).$$

For the final statement, it is clear that the expectation (A.1.6) is either finite for all $x > 0$ or for no $x > 0$, since the diffusion is regular. Now assuming (A.1.7) holds,

$$\int_0^\infty \left\{ \mathbb{E}^x \int_0^\infty e^{-\rho t}\Delta_t^\theta \, dt \right\} \pi(x)\, dx = \frac{1}{\rho}\, \mathbb{E}^\pi \Delta_0^\theta < \infty, \tag{A.1.8}$$

so the expectation (A.1.6) must be finite for all $x > 0$.

Conversely, if $\theta + 2A/\sigma^2 \leq 0$, then the expectation (A.1.8) is infinite. We prove by a coupling argument that this forces the expectation (A.1.6) to be also infinite. If we write Π for the invariant measure of Δ, and $\{P_t\}$ for its transition semigroup, then we have the ergodic result (see Rogers and Williams (2000), Section V.54)

$$\left\| \Pi - P_t(x, \cdot) \right\| \to 0, \qquad t \to \infty,$$

in the total variation norm for measures. This means that for each $\varepsilon > 0$, for each $x > 0$, one can find $T > 0$ such that $t > T$ ensures

$$|P_t(x, A) - \Pi(A)| < \varepsilon^{-2\theta},$$

simultaneously *for all* Borel subsets $A \subset \mathbb{R}^+$. By this, because $\theta < 0$, we have

$$\begin{aligned} \int_\varepsilon^\infty y^\theta |p_t(x,y) - \pi(y)|\, dy &\leq \int_\varepsilon^\infty \varepsilon^\theta |p_t(x,y) - \pi(y)|\, dy \\ &\leq \varepsilon^\theta \varepsilon^{-2\theta} = \varepsilon^{-\theta}. \end{aligned}$$

It is now clear that $\mathbb{E}^x \int_0^\infty e^{-\rho t}\Delta_t^\theta \, dt$ can be made arbitrarily large if (A.1.8) is infinite. ■

A.2. Excursion Computation of (III.4.50)

Here, we set up some notation and review some basic results from excursion theory that will allow us to compute the expression (III.4.50). For more details, see Rogers & Williams (2000) and Rogers (1989).

Let $W \equiv (W_t)_{t\geq 0}$ be a standard Brownian Motion, and denote the local time at 0 of W by the process $L \equiv (L_t)_{t\geq 0}$. Write $\overline{W} \equiv (\overline{W_t})_{t\geq 0} \equiv (\sup_{0\leq s\leq t} W_s)_{t\geq 0}$ for the supremum process of W, and let $\underline{W}$ be the corresponding infimum process. The processes L and $\overline{W}$ have points of increase that make up a set of Lebesgue measure zero, the former increasing when W hits 0 and the latter when W attains a new maximum. This fact points to a more subtle relation between the two processes, and it was Lévy who first proved the following identity in law.

Theorem A.2.1. *The process* $(\overline{W}_t, \overline{W}_t - W_t)_{t\geq 0}$ *(and, by symmetry,* $(-\underline{W}_t, W_t - \underline{W}_t)_{t\geq 0}$*) has the same bivariate law as* $(L_t, |W_t|)_{t\geq 0}$,

The *excursion space* U for W is defined to contain all continuous functions $f : \mathbb{R}^+ \to \mathbb{R}$ with $f^{-1}(\mathbb{R}/\{0\}) = (0,\zeta)$ for some $\zeta > 0$. Behind this definition is the desire to split any path of W into excursions of lifetimes ζ away from zero, and this can be accomplished because the set of points of increase of the local time process L (equivalently, $\overline{W}$) has no segment.

There is no ordering of the excursions in real time, but because the process $\overline{W}$ is non-decreasing and, by Lévy's result, *has the same law as the local time* L, a natural ordering can be imposed on the excursions using the local time process. We can therefore unambiguously speak of an excursion f made by W at local time t (that is, *an excursion of* W *downwards from an attained maximum* t); such pairs (t, f) can then be thought of as points of a Point Process Ξ. The crucial property of Ξ is that it is a Poisson Point Process (PPP) with measure Lebesgue $\times n$, where the *excursion measure* n is a σ-finite measure on U.

The importance of this result is that it translates (hard) questions about probabilities involving sample paths of W into (easy) questions about probabilities involving exponential random variables. For instance, if $C \subset U$, then the number of points of Ξ in $(0,t) \times C$ is Poisson distributed with mean $t\ n(C)$, so the local time when we first see an excursion in C is exponential with parameter $n(C)$.

We shall need the following

Proposition A.2.2. *Let $a > 0$, and define $C_a \equiv \{f \in U : \sup_t |f(t)| > a\}$. The excursion measure of C is $n(C) = 1/a$.*

Proof. By Levy's Theorem A.2.1, we have that $L - |W|$ is a Brownian motion. If we write $\tau_a = \inf\{t : W_t = a\}$, then the optional stopping theorem tells us that $\mathbb{E}L(\tau_a) = |W(\tau_a)| = a$. But $L(\tau_a)$ is the local time at zero when there is first an excursion in C, and this local time is exponential with parameter $n(C)$, which therefore equals $1/a$. ■

We now justify the expression (III.4.50).

Lemma A.2.3. *Let W be a standard Brownian Motion, U its excursion space, $\tilde{\mathbb{E}}$ its law. Then, for constants $r > 0$, $\theta \in \mathbb{R}$, we have*

$$\begin{aligned}\lambda(r,\theta) :&= n\left(\int_0^{\zeta(\xi)} e^{-ru-\theta|\xi_u|}du\right)\\ &\equiv \int_U \int_0^{\zeta(\xi)} e^{-ru-\theta|\xi_u|}du\ n(d\xi)\\ &= 2/(\sqrt{2r}+\theta),\end{aligned} \tag{A.2.1}$$

Here, the random variable ζ is the lifetime of the generic excursion ξ.

Proof. Given some number $y > 0$, write

$$C_y := \{f \in U : \sup_{0<t<\zeta} |f(t)| \geq y\}$$

for the set of excursions of $|W|$ that get above y, and set $H_y \equiv \inf\{t : |W_t| = y\}$.

Now, by monotone convergence,

$$
\begin{aligned}
n\left(\int_0^{\zeta} e^{-rt-\theta|\xi_t|}dt\right) &= \int_U \int_0^{\zeta(\xi)} e^{-rt-\theta|\xi_t|}dt\ n(d\xi) \\
&= \lim_{\varepsilon\downarrow 0} \int_{C_\varepsilon} \int_{H_\varepsilon}^{\zeta(\xi)} e^{-rt-\theta|\xi_t|}dt\ n(d\xi) \\
&= \lim_{\varepsilon\downarrow 0} \int_{C_\varepsilon} n(d\xi)\tilde{\mathbb{E}}^{\varepsilon}\left[\int_0^{H_0} e^{-rt-\theta|W_t|}dt\right] \\
&= \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\tilde{\mathbb{E}}^{\varepsilon}\left[\int_0^{H_0} e^{-rt-\theta|W_t|}dt\right].
\end{aligned}
\tag{A.2.2}
$$

The equality before last follows from the Markov property of excursions, which says that after having reached ε, the excursion ξ has the same law as brownian motion started at ε and killed at zero. This explains the appearance of the brownian expectation here, while Lemma A.2.2 gives us $n(C_\varepsilon)$ as $1/\varepsilon$.

From (III.4.18), we deduce the value of the integral in the last line above as

$$
\tilde{\mathbb{E}}^{\varepsilon} \int_0^{\tau_0} \exp\left(-rt - \theta W_t\right)dt = -2\frac{\left[e^{-\theta\varepsilon} - e^{-\sqrt{2r}\varepsilon}\right]}{\theta^2 - 2r}, \tag{A.2.3}
$$

which is finite for all θ. A simple limit calculation now yields (A.2.1). ■

A.3. The fluctuation identity (III.4.20)

We prove here the fluctuation identity employed in Section III.4.2. Alili and Kyprianou (2004) prove an equivalent identity by the same method, but they impose the more restrictive condition $\eta \geq 0$.

Proposition A.3.1. *Let X be a Lévy process, with Wiener-Hopf factors $\psi^{\pm}(\cdot)$, and set $\tau_x := \inf\{t : X_t < x\}$. Then provided $\psi^{-}(\eta) < \infty$, we have the Laplace transform*

$$
\int_{-\infty}^{0} \theta e^{\theta\xi}\ \mathbb{E}^0\left[\exp\left(-\delta\tau_\xi + \eta X(\tau_\xi)\right)\right] d\xi = 1 - \frac{\psi^{-}(\theta+\eta)}{\psi^{-}(\eta)}, \tag{A.3.1}
$$

where $\theta > 0$.

Proof. Fix $x < 0$ and let T, $\tilde{T}$ be two independent copies of an exponential random variable of rate δ, independent of X. Then

$$\begin{aligned}\mathbb{E}^0\Big[\exp(\eta \underline{X}_T);\ T > \tau_x\Big] &= \mathbb{E}^0\Big[\exp\Big(\eta\big(X(\tau_x) + \underline{X}(\tilde{T})\big)\Big);\ T > \tau_x\Big] \\ &= \psi^-(\eta)\mathbb{E}^0\Big[\exp(\eta X(\tau_x));\ T > \tau_x\Big] \\ &= \psi^-(\eta)\mathbb{E}^0\Big[\exp\big(\eta X(\tau_x) - \delta\tau_x\big)\Big], \qquad \text{(A.3.2)}\end{aligned}$$

where the first equality follows because of spatial homogeneity, the strong Markov property at τ_x, and the lack of memory of T. Independence of T and $\tilde{T}$ takes us from the first equality to the second, and the third equality is then obvious.

The left side of (A.3.2) is finite because

$$\mathbb{E}^0\Big[\exp(\eta \underline{X}_T);\ T > \tau_x\Big] \le \mathbb{E}^0\Big[\exp(\eta \underline{X}_T)\Big] = \psi^-(\eta),$$

which is finite by assumption. Taking Laplace transforms in (A.3.2) and changing the order of integration gives us

$$\int_{-\infty}^0 \theta e^{\theta x}\mathbb{E}^0\Big[\exp(\eta \underline{X}_T);\ T > \tau_x\Big]\, dx = \psi^-(\eta) - \psi^-(\eta + \theta),$$

which is the desired identity. ■

A.4. Multi-agent equilibrium with default

Chapter III considered the effect of a limited-liability assumption in a model where a single agent with exponential utility faces (unbounded) random dividend outputs from several firms and has the option to replace any of them at any time by zero future dividend. We have characterized (explicitly in the one dimensional case) the solution to the agent's optimal stopping problem, and its form is to accept a dividend cashflow until a specified lower barrier is breached.

It is natural to ask whether the solution we have computed for a single agent entitled to the aggregate output of the economy corresponds also to an equilibrium in an economy with several utility-maximizers each of whom is entitled to a fraction of the total output. Suppose we place ourselves in the one-firm ($N =$

1) multi-agent model ($J > 1$) described in Section III.2, and assume that the aggregate share amount available is either one or zero, that is, $\Phi \in \varphi^s$ and the share can be defaulted but not gradually downsized. Starting with the fraction $\theta_j > 0$ of the single share ($\sum_j \theta_j = 1$) available, agent j maintains a holdings process ϕ_j so as to maximize objective (III.2.1) (in continuous time) or (III.2.2) (in discrete time).

Now because we have placed no bound constraints on the process c_j^* attaining j's objective, it follows that *until default occurs*, agent j's state-price density is simply his marginal utility U_j' evaluated[1] at c_j^*. We shall endeavour to find a vector of constants $\underline{\lambda} = (\lambda_1, \ldots, \lambda_J)^T$, to be determined in terms of the initial distribution of the share among the agents, such that

$$\zeta := \lambda_j \zeta_j, \qquad \text{for all } j,$$

can be exhibited as a representative marginal utility and therefore acts as a state-price density for the market (see KLS 1990).

The problem that arises in the present context is the requirement that in equilibrium, all agents agree on the time τ when the firm is to default. In fact, this agreement will obtain for certain special choices of the model parameters, and we compute below the explicit form of the equilibrium with agents maximizing the objectives (III.2.1) and (III.2.2). In the general case, the ratios ζ_j/ζ are no longer constants, and the constants λ would need to be replaced by stochastic processes, leading to a form for the representative agent that is altogether more complicated[2].

[1]This principle is the basis of all that we did in Chapter II; the numerous references quoted there, e.g. Breeden (1979), KLS (1990), Aase (2002), explain it in several contexts.

[2]Basak and Cuoco (1998) study a two-agent economy where trading restrictions for one agent mean that a representative marginal utility cannot be defined unless one allows the weights λ_1, λ_2 to be stochastic.

A.4.1. The continuous-time case

Consider first the continuous-time objective (III.2.1). If at times $t < \tau$ prior to the time of default τ we define $\delta > 0$ and the function $U : \mathbb{R} \to \mathbb{R}$ through

$$\begin{aligned} e^{-\delta t} U(c(t)) \equiv e^{-\delta t} U(c(t); \underline{\lambda}) &= \max_{\sum_j c_j = c(t)} \sum_j \lambda_j U_j(c_j(t)) e^{-\delta_j t} \\ &= \max_{\sum_j c_j = c(t)} \sum_j -\lambda_j e^{-\delta_j t - \gamma_j c_j(t)}, \end{aligned} \tag{A.4.1}$$

in terms of the aggregate output $c(t)$ at time t and j's allocation $c_j(t)$, then we find explicitly,

$$e^{-\delta t} U'(c(t)) = K e^{-\delta t} e^{-\Gamma c(t)}, \tag{A.4.2}$$

whence

$$U(c(t)) = -\Gamma^{-1} K e^{-\Gamma c(t)}, \tag{A.4.3}$$

where $\Gamma^{-1} \equiv \sum_j \gamma_j^{-1} > 0$, $K \equiv K(\underline{\lambda}) = \exp(A)$ with $A = \Gamma \sum_j \gamma_j^{-1} \log(\lambda_j \gamma_j)$, and $\delta \equiv \Gamma \sum_j \gamma_j^{-1} \delta_j$. The function U, which apart from a positive multiplicative constant is of the same form that we used in Chapter III, is therefore a bona-fide representative utility for our market. The maximizing c_j^* in (A.4.1) are given by

$$\begin{aligned} \gamma_j c_j^*(t) &= [\log(\lambda_j \gamma_j / K)] + \Gamma c(t) + (\delta - \delta_j) t, \\ &= [\log(\lambda_j \gamma_j / K)] + \Gamma X_t + (\delta - \delta_j) t, \quad t < \tau, \end{aligned} \tag{A.4.4}$$

an expression *linear* in X which results from the market-clearing condition (III.2.3). Further, in (A.4.2) we have

$$\zeta_t(\underline{\lambda}) \equiv \zeta_t = K e^{-\delta t - \Gamma X_t}, \quad (t < \tau). \tag{A.4.5}$$

For the time of default τ to be well-defined, it is necessary and sufficient that the stopping times solving the individual agents' problems coincide, and it is not obvious that this happens in general. On the one hand, we know from the one-agent story that the representative agent would choose a critical level b^* satisfying

$$e^{-\Gamma b^*} = 1 + \Gamma / \beta^*$$

with β^* being the solution to $\psi(\delta) = \beta$; see Proposition III.4.1. On the other hand, agent j would want to default when the consumption process c_j^* in (A.4.4) hits a critical level, b_j^* say, satisfying

$$e^{-\gamma_j b_j^*} = 1 + \gamma_j/\beta_j^*$$

where β_j^* now solves $\psi_j(\delta_j) = \beta$, with $\psi_j(\cdot)$ being the Lévy exponent of the process c_j^*. From (A.4.4), therefore, we have agreement between default times if and only if it holds that

$$\gamma_j b_j^* = [\log(\lambda_j \gamma_j / K)] + \Gamma b^*.$$

This means that we need to have

$$\Big(1 + \gamma_j/\beta_j^*\Big) = \frac{K}{\lambda_j \gamma_j}\Big(1 + \Gamma/\beta^*\Big). \tag{A.4.6}$$

It is clear that only for one particular choice of the vector[3] $\underline{\lambda}$ will the displayed equality obtain[4]. For such a choice, a multi-agent equilibrium will exist. For the remainder of this section, we assume that an equilibrium exists in which the time of default τ is well-defined; this can be interpreted as either a one-agent market or as a multi-agent one in which (A.4.6) obtains.

Given τ is well-defined, we can use the state-price density ζ in (A.4.5) to deduce the equilibrium price process for the firm. At time 0, when the aggregate output is $X_0 = x \geq b^*$, the marginal price S_0 of cashflow X up to the time of default is given by

$$\begin{aligned}
\zeta_0 S_0 &= \mathbb{E}^x \Big[\int_0^{\tau(b^*)} \zeta_s X_s ds\Big] \\
e^{-\Gamma x} S_0 &= \mathbb{E}^x \Big[\int_0^{\tau(b^*)} e^{-\delta s - \Gamma X_s} X_s ds\Big] \\
&= -\frac{\partial}{\partial \theta}\mathbb{E}^x \Big[\int_0^{\tau(b^*)} e^{-\delta s - (\Gamma+\theta) X_s} ds\Big]\Big|_{\theta=0}.
\end{aligned} \tag{A.4.7}$$

Trivially, $S_0 = 0$ if $x < b^*$. The expected value on the right can be deduced from (III.4.18); explicitly, it is equal to

$$\frac{-e^{-\tilde{\Gamma} b^*}}{\delta - \psi_{BM}(-\tilde{\Gamma})}\Big\{e^{-\tilde{\Gamma}(x-b^*)} - e^{-\alpha(x-b^*)}\Big\} \tag{A.4.8}$$

[3]...and therefore, only one particular initial wealth distribution among agents...
[4] Some further simplification is possible for the Brownian case.

in the case when X is a Brownian motion as in Section III.4.3, and to

$$\frac{-e^{-\tilde{\Gamma}b^*}}{\delta-\psi_{CPP}(-\tilde{\Gamma})}\left\{e^{-\tilde{\Gamma}(x-b^*)}-\frac{c-\alpha^*}{c-\tilde{\Gamma}}e^{-\alpha^*(x-b^*)}\right\} \tag{A.4.9}$$

when X is a CPP of negative exponential jumps added to a drift; see expressions (III.4.27) and (III.4.32). Here we have written $\tilde{\Gamma} \equiv (\Gamma+\theta)$.

Differentiating in θ and evaluating at $\theta=0$, and using the Markov property of X, it results for the Brownian case that $S_t^{BM}=S^{BM}(X_t)$ where for $x \geq b^*$,

$$S^{BM}(x) := g^{BM}(x) - e^{-(\alpha^*-\Gamma)(x-b^*)}g^{BM}(b^*); \tag{A.4.10}$$

for the CPP X we get the analogous expression

$$S^{CPP}(x) := g^{CPP}(x) - \frac{c-\alpha^*}{c-\Gamma}e^{-(\alpha^*-\Gamma)(x-b^*)}g^{CPP}\Big(b^*-\frac{1}{c-\Gamma}\Big). \tag{A.4.11}$$

The function g^{BM} is defined by

$$g^{BM}(z) = z\kappa_{BM}(-\Gamma) - (\kappa_{BM}(-\Gamma))'; \qquad \kappa_{BM}(s) := \big(\delta-\psi_{BM}(s)\big)^{-1} \tag{A.4.12}$$

where the prime $'$ signifies differentiation in Γ. An exactly analogous definition holds for g^{CPP}, with ψ_{CPP} replacing ψ_{BM}.

We observed just before Proposition III.4.8 that (i) the assumption $\alpha^* > \Gamma$ ensures $\delta > \psi_{CPP}(-\Gamma)$, and that (ii) $c > \alpha^*$. The first of these facts implies that $g^{\cdot}$ is increasing, as can be seen directly from the definition of the functions $g^{\cdot}$. The remainder of the analysis now hinges on the sign of $g^{\cdot}(b^*)$, which turns out to be negative. To see this, note that the Wiener-Hopf factorization (III.4.7) and the characterization of b^* as the solution to (III.4.8) allow us to write

$$\delta\kappa(-\Gamma) = \psi^+(-\Gamma)\ \psi^-(-\Gamma) = e^{\Gamma b^*}\psi^-(-\Gamma).$$

By differentiating this expression in Γ we realise that $g^{\cdot}$ has the representation

$$g^{\cdot}(z) = \kappa(-\Gamma)(z-b^*) - (\psi^-(-\Gamma))'e^{\Gamma b^*}/\delta,$$

from which it is straightforward that $g^{\cdot}(b^*) \leq 0$.

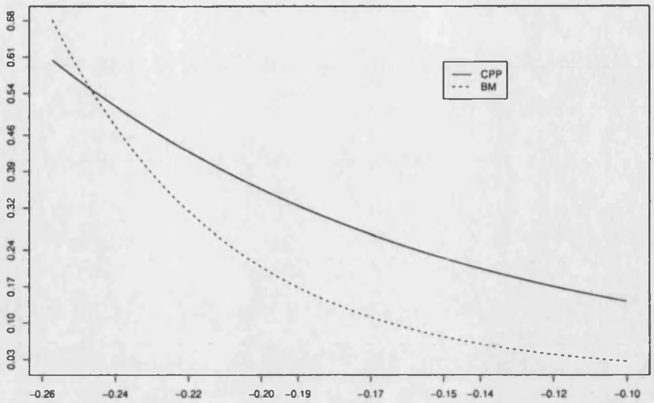


Figure A.1: Comparison of values of default option for BM and CPP dynamics with identical first and second moments. Levels of X are on the abscissa, values $S^{\cdot}(\cdot) - g^{\cdot}(\cdot)$ on the ordinate axis. For the CPP, we set $a = 0.5$, $c = 20$, $\mu_{CPP} = 0.05$. The BM has $\mu_{BM} = \mu_{CPP} - a/c$, $\sigma^2 = 2a/c^2$. Model parameters are $\delta = 0.08$, $\Gamma = 2$, leading to exercise boundaries $b^* = -0.266$ (CPP) and $b^* = -0.269$ (BM).

It now follows that the pricing functions $S^{\cdot}$ are convex, and also that the functions

$$S^{\cdot}(\cdot) - g^{\cdot}(\cdot) \tag{A.4.13}$$

are decreasing and positive. Why this should be so is intuitive. As we let $b^* \downarrow -\infty$ and keep all other parameters unchanged, we find for each $x > b^*$ that $S^{\cdot}(x) \to g^{\cdot}(x)$. The function $g^{\cdot}$ is therefore the pricing function in a market where the agents have not the option to default the firm. For each x, $S^{\cdot}(x) - g^{\cdot}(x)$ must then be the value of the option to default, and this we *do* expect to be non-negative and decreasing in x.

Figure A.1 compares the default option value of a CPP with that of a Brownian motion having identical mean and variance. It is worth noting that the decay is much slower for the jump process than for the diffusion. This is to be expected, because the occurence of a jump may well cause default even when the process X is well above b^*. The same reasoning explains why the critical level b^* is slightly higher for the jump process.

A.4.2. The discrete-time case

The procedure we followed above can be repeated in the discrete-time setting of Section III.3, where the output process X was an IID sequence $\{(X_k), k \in \mathbb{Z}^+\}$ of Gaussian random variables distributed as $N(\mu, \sigma^2)$. Recall also from Section III.2 that β_j, $1 \leq j \leq J$, is the discount factor for agent j that appears in the objective (III.2.2).

We do not go again into issues regarding the well-definition of the default time τ, as the points to check are bound to be similar to those of the continuous-time setting. Assume that an equilibrium exists where all agents agree on the time of default. A representative utility U and discount rate β can then be defined for $n < \tau$, $n \in \mathbb{Z}^+$, through

$$\begin{aligned} \beta^n U(c(n)) \equiv \beta^n U(c(n); \underline{\lambda}) &= \max_{\sum_j c_j = c(n)} \sum_j \lambda_j \beta_j^n U_j(c_j(n)) \\ &= \max_{\sum_j c_j = c(n)} \sum_j -\lambda_j \beta_j^n e^{-\gamma_j c_j(n)}, \end{aligned} \tag{A.4.14}$$

in terms of the aggregate output $c_j(n)$ at time period n and the chunk $c_j(n)$ of it that is taken by agent j. Then, as in Section A.4.1, we get that

$$\beta^n U'(c(n)) \equiv \zeta_n = K\beta^n e^{-\Gamma c(n)}, \tag{A.4.15}$$

where K and Γ are as defined after (A.4.3) and $\log \beta \equiv \Gamma \sum_j \log{(\beta_j)}/\gamma_j$. The maximizing c_j^* in (A.4.14) now satisfy

$$\gamma_j c_j^*(n) = \Gamma c(n) + \log{(\lambda_j \gamma_j / K)} + \log{(\beta_j^n / \beta^n)}. \tag{A.4.16}$$

The aggregate output available at time n is of course X_n, so in (A.4.15) we get

$$\zeta_n = K\beta^n e^{-\Gamma X_n}, \qquad n < \tau. \tag{A.4.17}$$

Because the representative utility is of the same form as in the one-agent model we described in Section III.3, we know that the time of default is $\tau = \tau(b^*)$, with the critical level b^* being determined from (III.3.5). In similar fashion to what we did in Section A.4.1, we can now derive the marginal price of output X up to the time of default in this equilibrium.

From (A.4.17), the time-0 marginal price S_0^{IID} of the cashflow X up to the time of default, given that $X_0 = x \geq b^*$, satisfies

$$
\begin{aligned}
e^{-\Gamma x} K S_0^{IID} &= \mathbb{E}\Big[\sum_{n=0}^{\tau(b^*)-1} \zeta_n X_n \Big| X_0 = x\Big] \\
e^{-\Gamma x} S_0^{IID} &= x e^{-\Gamma x} + \mathbb{E}\Big[\sum_{n=1}^{\infty} \beta^n X_n e^{-\Gamma X_n} \mathbf{1}\big(n < \tau_{b^*}\big)\Big] \\
&= x e^{-\Gamma x} + \mathbb{E}\Big[\sum_{n=1}^{\infty} [\beta^n X_n e^{-\Gamma X_n} \quad ; X_n \geq b^*] \mathbb{P}\big(X_1 \geq b^*\big)^{n-1}\Big]. \qquad \text{(A.4.18)}
\end{aligned}
$$

Clearly, $S_0^{IID} = 0$ if $X_0 < b^*$.

Seeing that the X_n are IID $\sim N(\mu, \sigma^2)$, the expectation above is computed easily, and because there is nothing special about the time 0 here, in the end we get the asset price as a function $S_k^{IID} \equiv S^{IID}(X_k)$, $k \in \mathbb{Z}^+$, with

$$
S^{IID}(x) = \begin{cases} x + \frac{\beta A}{1 - \beta \bar{\Phi}(z)} e^{\Gamma x}, & x \geq b^*, \\ 0 & x < b^*, \end{cases} \qquad \text{(A.4.19)}
$$

where $z = (b^* - \mu)/\sigma$, $A = \int_{b^*}^{\infty} x e^{-\Gamma x} \phi((x - \mu)/\sigma) dx/\sigma$, ϕ is the standard normal density, and $\Phi \equiv 1 - \bar{\Phi}$ the cumulative standard normal distribution.

The price process S^{IID} here incorporates the value of the observed output x, together with the net present value of future output and the option that the agents have to default. The value of the latter can be deduced by subtracting the limiting value as $b^* \downarrow -\infty$, exactly as in the continuous-time case.

A.4.3. The individual agent's wealth

Because we have computed the form of the processes c_j^* attaining agents' objectives and also the state-price density ζ, we can now show how the vector of equilibrium weights $\underline{\lambda}$ can be chosen so that agent j's cashflow process c_j^* has an equilibrium price equal to that of the positive amount of shares θ_j that j starts with at time 0.

Again, we treat the continuous-time case; the discrete-time version is dealt with in exactly analogous manner. If c_j^* is the process, given in (A.4.4), that attains j' objective, then we require that the marginal price of this cashflow coincide with the market value of j's initial share amount:

$$\zeta_0 \theta_j S_0 = \mathbb{E}^x \left[\int_0^\tau \zeta(s) c_j^*(X_s)\, ds \right].$$

The expectation is in the law of X, which starts at $x = X_0$, and we have made explicit that the allocation c_j^* is a function of X. Recall also that here $\tau = \inf\{t : X_t < b^*\}$. Now choose a normalization $K = 1$, which entails the condition that the equilibrium vector $\underline{\lambda} = (\lambda_1, \ldots, \lambda_J)^T$ satisfy

$$\sum_j \gamma_j^{-1} \log(\lambda_j \gamma_j) = 0;$$

note that different choices of K affect the state price density (A.4.5) only through a scaling constant, and hence leave all marginal prices unchanged. If we express c_j^* in terms of X from (A.4.4), we change the expectation displayed above into

$$\begin{aligned} \zeta_0 \gamma_j \theta_j S_0 = \log(\lambda_j \gamma_j) \mathbb{E}^x \left[\int_0^\tau \zeta_s ds \right] \\ + \Gamma \tilde{\mathbb{E}}^y \left[\int_0^{\tau(b^* + y - x)} Y_t e^{-\delta_j t - \Gamma Y_t}\, dt \right], \end{aligned} \qquad \text{(A.4.20)}$$

where $Y_t := X_t + \bar{\delta}/\Gamma$ with $y = Y_0 = x + \bar{\delta}/\Gamma$, $\bar{\delta} \equiv \delta - \delta_j$, and $\tilde{\mathbb{E}}$ is expectation in the law of Y. The constant λ_j, which determines completely j's equilibrium share of the output X, can now be chosen to solve (A.4.20); the expectations involved are of the same general form as in (A.4.7) and in (III.4.18).

A.5. Tables of Results

Table A.1: MC estimates of quantities of interest for several simulated model parameters, with $\mu_k > 0$, $k = 1, 2$. Sample size was 20000, and antithetic variables were used. NI denotes values obtained from numerical integration on a 400×400 grid of values for $(X_1, X_2)^T$, with grid spacing 0.03.

Simulated Parameters					MC estimation				NI estimation			
σ_1	μ_1	ρ	β	d_1	p_1	p_{12}	$V_{\mathbb{D}^*}$	K	p_1	p_{12}	$V_{\mathbb{D}^*}$	K
σ_2	μ_2		Γ	d_2	p_2	q	$p_{\mathbb{D}^*}$		p_2	q	$p_{\mathbb{D}^*}$	
0.555	0.232	-0.718	0.892	0.079364	0.03935	0.00005	0.001817	-7.736278	0.038395	0.000036	0.00031	-7.723136
0.168	0.017		1.223	0.167973	0.0014	0.9592	0		0.001105	0.960463	0	
0.86	0.292	-0.778	0.898	0.216766	0.02835	0.00075	0.028368	-6.480615	0.031144	0.000972	0.021838	-6.551996
0.639	0.152		1.903	0.23602	0.00505	0.96585	0		0.006449	0.961434	0	
0.492	0.12	-0.991	0.997	0.014847	0	0	0.014626	-211.154273	0	0	0.013749	-211.453804
0.678	0.382		1.022	0	0	1	0		0	1	0	
0.977	0.191	-0.545	0.953	0.000106	0.0001	0	0.000015	-21.06837	0.000196	0	0	-21.077191
0.312	0.178		0.027	0	0	0.9999	0		0	0.999804	0	
0.641	0.398	-0.746	0.953	0.008913	0	0	0.003974	-11.799304	0	0	0.00382	-11.784508
0.943	0.313		1.33	0.162998	0.0003	0.9997	0		0.000483	0.999517	0	
0.956	0.218	-0.415	0.987	0.126884	0.0281	0.00325	0.002151	-73.604336	0.027468	0.003233	0	-73.664933
0.767	0.128		0.819	0.137863	0.0172	0.95145	0.00005		0.01717	0.95213	0	
0.503	0.162	-0.409	0.969	0.018966	0.00005	0.00005	0.008341	-27.411171	0.000169	0.000051	0.007023	-27.454803
0.622	0.178		0.99	0.065959	0.00145	0.99845	0		0.00164	0.99814	0	
0.393	0.029	-0.716	0.996	0.214712	0	0	0.144094	-176.480643	0	0	0.145986	-176.054868
0.492	0.237		1.663	0.000092	0	1	0		0	1	0	
0.365	0.16	-0.951	0.954	0.174552	0	0	0.013776	-2.106801	0	0	0.013756	-2.107284
0.322	0.3		5.504	0.020575	0	1	0		0	1	0	
0.455	0.055	0.091	0.936	0.040572	0.0338	0.0063	0.000063	-15.505746	0.036333	0.005263	0	-15.507219
0.357	0.017		0.106	0.108462	0.1091	0.8508	0		0.108805	0.849599	0	
0.681	0.051	-0.625	0.961	0.279788	0.00595	0.00045	0.023572	-19.861429	0.006003	0.000148	0.022528	-19.88933
0.695	0.373		1.388	0.029046	0.00045	0.99315	0		0.00046	0.993388	0.000004	
0.703	0.179	-0.94	0.997	0.230041	0	0	0.209856	-253.367117	0	0	0.207164	-254.25058
0.831	0.028		2.465	0.373243	0	1	0.0001		0	1	0.00004	
0.625	0.147	-0.229	0.949	0.000002	0	0	0.000011	-19.493498	0.000002	0	0	-19.493645

Table A.1: (continued)

Parameters					MC				NI			
σ_1 σ_2	μ_1 μ_2	ρ	β Γ	d_1 d_2	p_1 p_2	p_{12} q	$V_{\mathbb{D}^*}$ $p_{\mathbb{D}^*}$	K	p_1 p_2	p_{12} q	$V_{\mathbb{D}^*}$ $p_{\mathbb{D}^*}$	K
0.648	0.181		0.001	0	0	1	0		0	0.999998	0	
0.527	0.093	-0.698	0.965	0.002012	0	0	0.001916	-27.455788	0	0	0.000937	-27.470724
1.229	0.151		0.228	0.093375	0.02255	0.97745	0		0.022866	0.977134	0	
0.69	0.219	-0.744	0.986	0.082319	0	0	0.049261	-51.51766	0	0	0.049915	-51.469682
0.439	0.194		1.242	0.000022	0	1	0		0	1	0	

Table A.2: Monte Carlo estimates of quantities of interest for several simulated model parameters, with $\mu_k > 0$, $k = 1, 2$. Sample size was 20000, and antithetic variables were used.

Simulated Parameters					Estimated values			
σ_1	μ_1	ρ	β	d_1	p_1	p_{12}	V_{D^*}	K
σ_2	μ_2		Γ	d_2	p_2	q	p_{D^*}	
0.733	0.263	-0.726	0.918	0.08383	0.00065	0	0.036518	-9.164598
0.641	0.167		0.899	0.122073	0.0004	0.99895	0.0001	
0.356	0.036	-0.543	0.985	0.000004	0	0	0.000271	-68.156413
0.202	0.022		0.161	0	0	1	0	
0.73	0.199	-0.463	0.907	0.009491	0.0034	0	0.000111	-10.362938
0.569	0.162		0.116	0.005241	0.0014	0.9952	0	
0.442	0.065	-0.256	0.993	0	0	0	0.000442	-146.010298
0.901	0.073		0.099	0.000005	0.00405	0.99595	0	
0.731	0.093	-0.516	0.963	0.001114	0.0006	0	0.000003	-26.86528
0.35	0.147		0.032	0	0	0.9994	0	
0.658	0.295	-0.791	0.995	0.000004	0	0	0.12426	-152.382099
0.529	0.137		1.05	0.02629	0	1	0	
0.627	0.196	-0.438	0.988	0.023847	0	0	0.074893	-73.109842
0.681	0.21		0.985	0.039949	0	1	0	
0.536	0.135	-0.725	0.861	0.142627	0.0038	0	0.000128	-5.319946
0.674	0.317		0.837	0.060521	0.0026	0.9936	0	
0.408	0.184	-0.775	0.971	0.006368	0	0	0.068129	-20.342546
0.826	0.389		1.605	0.091291	0.0002	0.9998	0	
0.852	0.096	-0.749	0.986	0.024808	0	0	0.004348	-67.699497
0.362	0.059		0.221	0	0	1	0	
0.434	0.049	-0.66	0.991	0.086216	0	0	0.042338	-98.35301
0.669	0.27		0.776	0.000001	0	1	0	
0.744	0.118	-0.499	0.939	0.023312	0.0117	0	0.000236	-16.014706
0.638	0.068		0.105	0.057536	0.02985	0.95845	0	
0.47	0.157	-0.984	0.997	0.098677	0	0	0.541627	-122.682974
0.552	0.354		2.235	0.001565	0	1	0	
0.697	0.156	-0.815	0.967	0.243387	0.00255	0	0.068195	-20.66598
0.414	0.213		2.192	0.023133	0	0.99745	0	
0.586	0.179	-0.929	0.865	0.190832	0.0123	0	0.001837	-4.693078
0.337	0.166		1.672	0.051338	0	0.9877	0	
0.499	0.079	-0.87	0.992	0.219497	0	0	0.383286	-73.605751
0.659	0.312		2.086	0.078351	0	1	0	
0.829	0.148	-0.671	0.984	0	0	0	0.000143	-61.332165
0.412	0.052		0.14	0	0	1	0	
0.754	0.205	-0.564	0.991	0.183142	0.01275	0.00355	0.013552	-111.65164
0.624	0.158		1.759	0.163612	0.0052	0.9785	0.00025	
0.343	0.126	-0.708	0.988	0.000485	0	0	0.01742	-71.013344
0.622	0.114		1.604	0.198412	0.0012	0.9988	0	
0.621	0.232	-0.895	0.996	0.015512	0	0	0.256285	-178.419756
0.635	0.086		1.246	0.191807	0	1	0	
0.688	0.053	-0.909	0.993	0.315068	0	0	0.191688	-104.65467
0.547	0.154		2.119	0.153695	0	1	0	
0.546	0.122	-0.842	0.976	0.011437	0	0	0.054436	-36.240238
0.889	0.166		0.584	0.103973	0	1	0	
0.502	0.107	-0.498	0.948	0.118177	0.00475	0.0001	0.001998	-16.117145
0.496	0.19		1.068	0.023467	0.00075	0.9944	0	
0.648	0.092	-0.402	0.937	0.012129	0.01145	0	0.000001	-15.887128

Table A.2: (continued)

Simulated Parameters					Estimated values			
σ_1 / σ_2	μ_1 / μ_2	ρ	β / Γ	d_1 / d_2	p_1 / p_2	p_{12} / q	V_{D^*} / p_{D^*}	K
0.407	0.214		0.012	0	0	0.98855	0	
0.578	0.096	-0.615	0.968	0.000968	0	0	0.000425	-29.669651
0.888	0.236		0.18	0.000012	0	1	0	
0.696	0.192	-0.51	0.974	0.11538	0.001	0.0003	0.048632	-32.019851
0.645	0.197		1.184	0.083617	0.0003	0.9984	0.00025	
0.694	0.229	-0.603	0.988	0.008451	0	0	0.106213	-67.426481
0.788	0.195		0.842	0.082454	0	1	0.00005	
0.593	0.127	-0.951	0.999	0.005063	0	0	0.138805	-675.285907
0.659	0.104		0.722	0.082254	0	1	0	
0.67	0.112	-0.594	0.985	0.147323	0.0037	0.0012	0.011906	-62.37172
0.718	0.125		0.978	0.152718	0.00585	0.98925	0.0002	
0.625	0.209	-0.766	0.942	0.145798	0.0007	0	0.032363	-10.551646
0.418	0.221		1.726	0.016742	0	0.9993	0	
0.864	0.056	-0.963	0.981	0.141306	0	0	0.037163	-48.327889
0.778	0.147		0.357	0.010607	0	1	0	
0.713	0.168	-0.586	0.897	0.032693	0.0095	0	0.00011	-9.306849
0.942	0.166		0.141	0.079264	0.04485	0.94565	0	
0.387	0.11	-0.966	0.955	0.036302	0	0	0.043387	-18.742843
0.813	0.159		1.109	0.196199	0.0031	0.9969	0	
0.584	0.265	-0.938	0.982	0.088196	0	0	0.419049	-23.930625
0.455	0.161		2.245	0.096054	0	1	0	
0.686	0.15	-0.738	0.972	0.001555	0	0	0.00613	-31.28124
0.93	0.28		0.316	0.000295	0	1	0	
0.846	0.259	-0.862	0.886	0.060308	0.02055	0	0.000055	-8.00065
0.492	0.021		0.355	0.210902	0.0426	0.93685	0	
0.391	0.059	-0.698	0.987	0.034859	0	0	0.03118	-68.977769
0.401	0.1		0.747	0.000019	0	1	0	
0.718	0.132	-0.21	0.999	0.093196	0.0087	0.00005	0.003063	-709.973292
0.654	0.161		0.799	0.017728	0.00115	0.9901	0.00005	
0.479	0.047	-0.298	0.996	0.124662	0.0059	0	0.001725	-260.845278
0.502	0.116		0.86	0.0023	0.00015	0.99395	0	
0.889	0.152	-0.453	0.964	0.003178	0.00085	0	0.000938	-27.585693
0.638	0.033		0.137	0.083104	0.0344	0.96475	0	
0.638	0.208	-0.574	0.97	0.027835	0	0	0.054918	-27.21083
0.676	0.172		0.869	0.078824	0	1	0	
0.703	0.212	-0.475	0.944	0.063164	0.0046	0	0.000035	-15.061474
0.544	0.167		0.752	0.032513	0.0005	0.9949	0	
0.43	0.062	-0.449	0.971	0.035738	0.00175	0.0001	0.003032	-33.655526
0.464	0.053		0.495	0.070861	0.0063	0.99185	0	
0.77	0.11	-0.54	0.984	0.002046	0	0	0.000241	-58.686543
0.822	0.203		0.226	0	0	1	0	
0.762	0.263	-0.781	0.947	0.015084	0	0	0.033803	-15.674916
0.619	0.147		0.532	0.044544	0	1	0	
0.709	0.146	-0.705	0.996	0.123772	0	0	0.108782	-221.868305
0.51	0.112		1.038	0.049669	0	1	0	
0.86	0.221	-0.67	0.993	0.130999	0	0	0.077556	-125.113604
0.849	0.14		1.08	0.195001	0	1	0.00095	

Table A.3: Monte Carlo estimates of quantities of interest for several simulated model parameters, with $\mu_1 > 0$, $\mu_2 < 0$. Sample size was 20000, and antithetic variables were used.

Simulated Parameters					Estimated values			
σ_1	μ_1	ρ	β	d_1	p_1	p_{12}	V_{D^*}	K
σ_2	μ_2		Γ	d_2	p_2	q	p_{D^*}	
0.641	-0.611	-0.796	0.976089	0.802265	0.7956	0.0011	0.000003	-41.483463
0.866	0.101		0.164166	0.019668	0.01155	0.19175	0	
0.073	-0.250	0.928	0.966962	0.999711	0.8559	0.1401	0.000036	-28.372969
0.673	0.293		2.321157	0.134559	0	0.004	0	
0.650	-0.234	0.138	0.889024	0.505903	0.50915	0.0002	0.000015	-8.94195
0.695	0.272		0.026264	0.000393	0.0002	0.49045	0	
0.184	-0.478	0.191	0.98304	0.995308	0.99345	0.0001	0.000144	-42.324722
0.392	0.323		2.427996	0.000067	0	0.00645	0	
0.130	-0.247	0.332	0.961886	0.970399	0.7916	0.17195	0.000092	-24.820525
0.368	0.124		3.829185	0.172425	0.0007	0.03575	0	
0.096	-0.443	-0.075	0.982067	0.999998	0.91535	0.02765	0.000152	-45.293016
0.563	0.585		4.775441	0.026238	0	0.057	0	
0.952	-0.089	-0.859	0.984614	0.205277	0.1987	0	0.000002	-64.932584
0.280	0.029		0.021779	0	0	0.8013	0	
0.123	-0.341	0.083	0.962147	0.997259	0.99675	0	0.000004	-24.690087
0.339	0.376		0.185063	0	0	0.00325	0	
0.480	-0.288	0.067	0.796032	0.665727	0.5291	0.1348	0.000053	-4.838793
0.476	0.055		0.112807	0.190347	0.05555	0.28055	0	
0.240	-0.926	0.344	0.932457	0.999945	0.99995	0	0.000096	-10.988885
0.225	0.366		0.86626	0	0	0.00005	0	
0.294	-0.188	-0.040	0.952621	0.667973	0.66035	0	0.000003	-20.976786
0.207	0.089		0.068026	0	0	0.33965	0	
0.309	-0.518	0.675	0.962426	0.951296	0.949	0	0.000002	-26.5442
0.603	0.156		0.017128	0	0	0.051	0	
0.401	-0.954	0.435	0.98928	0.991315	0.9167	0.0715	0.000028	-91.17842
0.757	0.299		1.377898	0.070205	0	0.0118	0	
0.237	-0.366	0.039	0.973604	0.93528	0.9348	0	0.000001	-37.264712
0.271	0.062		0.33292	0	0	0.0652	0	
0.082	-0.495	-0.871	0.994965	1	1	0	0.000011	-144.082581
0.436	0.412		1.01929	0	0	0	0	
0.780	-0.220	0.807	0.923929	0.43512	0.2994	0.1431	0.000011	-13.110486
1.217	0.021		0.020315	0.149433	0.0077	0.5498	0	
0.373	-0.086	-0.684	0.889433	0.408094	0.4035	0	0.000004	-8.969687
0.140	0.146		0.04931	0	0	0.5965	0	
0.151	-0.345	-0.505	0.937899	0.988508	0.98875	0	0.000017	-14.765743
0.382	0.304		0.308381	0	0	0.01125	0	
0.405	-0.175	-0.697	0.911039	0.553302	0.54535	0.0029	0.000002	-11.11358
0.424	0.062		0.155868	0.062202	0.0515	0.40025	0	
0.290	-0.397	-0.078	0.871123	0.909476	0.63605	0.26935	0.000201	-7.477614
0.382	0.000		1.055536	0.299924	0.031	0.0636	0	
0.812	-0.127	0.895	0.938085	0.311252	0.3143	0	0.000004	-16.12206
0.450	0.283		0.005687	0	0	0.6857	0	
0.212	-0.020	0.929	0.972385	0.211797	0.2137	0	0.000028	-36.116641
0.722	0.116		0.022836	0	0	0.7863	0	
0.368	-0.761	0.514	0.949983	0.980384	0.98145	0	0.000006	-19.659337
0.312	0.391		0.043316	0	0	0.01855	0	
0.354	-0.606	-0.549	0.983354	0.954801	0.89125	0.0633	0.000003	-59.210649

Table A.3: (continued)

Simulated Parameters					Estimated values			
σ_1	μ_1	ρ	β	d_1	p_1	p_{12}	$V_{\mathbb{D}^*}$	K
σ_2	μ_2		Γ	d_2	p_2	q	$p_{\mathbb{D}^*}$	
0.549	0.109		0.929213	0.080919	0.0147	0.03075	0	
0.366	-0.226	-0.090	0.956139	0.655698	0.5777	0.0748	0.000012	-22.714989
0.423	0.006		0.140502	0.125104	0.0452	0.3023	0	
0.293	-0.392	-0.668	0.993364	0.903062	0.89525	0.00755	0.000001	-149.820502
0.322	0.053		1.025294	0.021678	0.0083	0.0889	0	
0.425	-0.712	0.225	0.984734	0.95113	0.82465	0.1263	0.000006	-65.28576
0.500	0.011		0.341724	0.126243	0.0022	0.04685	0	
0.237	-0.443	-0.530	0.885669	0.968401	0.968	0	0.000019	-8.677047
0.291	0.137		0.059148	0.000041	0.00005	0.03195	0	
0.290	-0.426	0.411	0.917804	0.924788	0.9117	0.0133	0.000001	-12.135793
0.220	0.041		0.059602	0.012837	0.0001	0.0749	0	
0.234	-0.018	-0.993	0.942248	0.334666	0	0	0.099114	-12.869197
0.493	0.232		2.037718	0.070322	0.0003	0.9997	0	
0.366	-0.072	-0.041	0.909932	0.367115	0.34765	0.021	0.000003	-11.096975
0.292	0.032		0.008766	0.066863	0.0443	0.58705	0	
0.149	-0.707	-0.218	0.952377	0.999999	1	0	0.000081	-19.048355
0.341	0.172		0.767242	0.000003	0	0	0	
0.398	-0.061	-0.526	0.967363	0.291721	0.28455	0	0.000002	-30.548758
0.375	0.048		0.056243	0.000275	0.0001	0.71535	0	
0.120	-0.447	0.546	0.962879	0.999904	0.6924	0.294	0.000017	-25.441501
0.616	0.198		6.329717	0.290832	0	0.0136	0	
0.438	-0.074	-0.932	0.990775	0.353925	0	0	0.000852	-90.546253
0.688	0.385		0.649279	0	0	1	0	
0.372	-0.297	-0.221	0.837752	0.750728	0.72105	0.0289	0.000054	-6.068711
0.338	0.097		0.148273	0.046096	0.0176	0.23245	0	
0.492	-0.113	-0.311	0.927739	0.392714	0.36285	0.02275	0.00001	-13.74309
0.680	0.039		0.084397	0.116802	0.09285	0.52155	0	
0.428	-0.321	0.890	0.975022	0.719042	0.71845	0	0.000001	-40.000967
0.898	0.100		0.008809	0.000012	0	0.28155	0	
0.182	-0.241	-0.936	0.938125	0.90008	0.89655	0	0.000098	-14.895316
0.472	0.200		0.620478	0.001028	0.0003	0.10315	0	
0.169	-0.185	0.635	0.927278	0.846168	0.84085	0.0009	0.000046	-13.098992
0.571	0.212		0.295586	0.000777	0	0.15825	0	
0.093	-0.289	-0.144	0.966616	0.999007	0.81215	0.1865	0.000181	-29.37305
0.966	0.112		0.588502	0.184035	0.0006	0.00075	0	
0.796	-0.318	-0.037	0.990731	0.504549	0.4991	0.0079	0.000001	-107.864693
0.713	0.012		0.010552	0.017303	0.00975	0.48325	0	
0.088	-0.503	-0.985	0.953068	1	1	0	0.000024	-19.27007
0.170	0.263		0.390519	0	0	0	0	
0.272	-0.549	0.603	0.811338	0.97768	0.7149	0.26175	0.00021	-5.146102
0.380	0.019		0.517708	0.259268	0.0002	0.02315	0	
0.180	-0.396	0.791	0.924158	0.986035	0.9836	0.0015	0.000067	-12.893812
0.340	0.102		0.257316	0.001264	0	0.0149	0	
0.287	-0.314	0.342	0.896855	0.847022	0.74455	0.0998	0.000026	-9.638175
0.783	0.084		0.05324	0.100976	0.00355	0.1521	0	
0.171	-0.329	-0.376	0.914957	0.972405	0.97025	0	0.00001	-11.207575
0.732	0.385		0.137743	0.000002	0	0.02975	0	

Table A.3: (continued)

Simulated Parameters					Estimated values			
σ_1 σ_2	μ_1 μ_2	ρ	β Γ	d_1 d_2	p_1 p_2	p_{12} q	$V_{\mathbb{D}^*}$ $p_{\mathbb{D}^*}$	K
0.159 0.832	-0.680 0.174	0.458	0.999505 0.320962	0.999991 0	1 0	0 0	0.000001 0	-1980.946315
0.418 0.482	-0.524 0.008	-0.067	0.990417 0.61398	0.885745 0.176763	0.73 0.0204	0.1553 0.0943	0.000006 0	-104.021837
0.370 0.208	-0.649 0.045	-0.442	0.882884 0.09524	0.959113 0.032346	0.93405 0.0057	0.02605 0.0342	0.00001 0	-8.50079
0.367 0.252	-0.212 0.015	-0.545	0.974809 0.041938	0.630095 0.010257	0.61895 0.0081	0.00095 0.372	0.000005 0	-39.667662
0.202 0.294	-0.604 0.035	0.063	0.944753 0.405474	0.998578 0.058494	0.9396 0.00005	0.0589 0.00145	0.000053 0	-17.901754
0.086 0.189	-0.310 0.388	-0.380	0.964317 0.441108	0.999843 0	0.9998 0	0 0.0002	0.000004 0	-23.698072
0.227 0.325	-0.211 0.259	0.313	0.752302 0.470996	0.80259 0.004634	0.7949 0.00025	0.0048 0.20005	0.000024 0	-3.604125
0.566 0.385	-0.023 0.175	-0.910	0.940782 2.580122	0.399825 0.076858	0.0721 0	0 0.9279	0.006246 0	-13.542312
0.198 0.762	-0.431 0.077	-0.222	0.986835 1.039419	0.98505 0.224428	0.76575 0.00605	0.22015 0.00805	0.000031 0	-75.269382
0.180 0.185	-0.455 0.126	0.590	0.980953 0.571605	0.994139 0	0.9951 0	0 0.0049	0.00002 0	-49.116985
0.283 0.240	-0.656 0.320	0.104	0.957896 0.13116	0.989673 0	0.9907 0	0 0.0093	0.000005 0	-22.785721
0.847 0.980	-0.946 0.076	0.270	0.90294 0.110207	0.853555 0.145298	0.72305 0.00895	0.13385 0.13415	0.000052 0	-10.173012
0.251 0.573	-0.004 0.165	-0.890	0.942818 0.849654	0.228442 0.061916	0 0.00675	0 0.99325	0.014351 0	-15.989434
0.550 0.984	-0.040 0.408	-0.735	0.950812 0.499454	0.289722 0.008631	0.0015 0.0002	0 0.9983	0.007773 0	-17.933896
0.282 0.566	-0.353 0.361	0.176	0.936463 0.006498	0.884845 0	0.8825 0	0 0.1175	0.000001 0	-15.702014
0.107 0.348	-0.249 0.187	-0.666	0.943753 0.246971	0.989854 0	0.99 0	0 0.01	0.000023 0	-17.039793
0.104 0.209	-0.266 0.199	0.259	0.936672 1.757735	0.994608 0	0.9951 0	0 0.0049	0.00014 0	-11.90028
0.812 0.855	-0.094 0.495	-0.882	0.970518 1.347259	0.414983 0.031108	0 0	0 1	0.157013 0	-22.954465
0.787 0.422	-0.096 0.187	-0.537	0.856792 0.002628	0.329578 0.000992	0.32805 0.00075	0 0.6712	0.000003 0	-6.978025
0.169 0.173	-0.659 0.049	-0.927	0.982221 0.314622	0.999952 0	0.99995 0	0 0.00005	0.000001 0	-55.460593
0.527 0.584	-0.886 0.298	-0.143	0.937786 0.110329	0.951783 0	0.95405 0	0 0.04595	0.000011 0	-15.584098
0.630 0.778	-0.266 0.105	0.165	0.954899 0.010698	0.531194 0.001891	0.5328 0.00055	0.0017 0.46495	0.000003 0	-22.14595
0.060 0.579	-0.205 0.124	-0.375	0.954193 0.223742	0.999691 0.002478	0.997 0	0.0028 0.0002	0.000047 0	-21.411096
0.324	-0.291	0.993	0.937951	0.783112	0.66145	0.1294	0.000009	-16.088658

Table A.3: (continued)

Simulated Parameters					Estimated values			
σ_1	μ_1	ρ	β	d_1	p_1	p_{12}	$V_{\mathbb{D}^*}$	K
σ_2	μ_2		Γ	d_2	p_2	q	$p_{\mathbb{D}^*}$	
0.686	0.011		0.0324	0.132569	0	0.20915	0	
0.184	-0.440	0.379	0.907151	0.991496	0.94555	0.0447	0.000047	-10.52368
1.004	0.220		0.131735	0.0406	0.00005	0.0097	0	
0.345	-0.504	-0.020	0.955043	0.923688	0.8846	0.03915	0.000011	-22.152336
0.331	0.029		0.150607	0.041716	0.00375	0.0725	0	
0.641	-0.039	0.905	0.896446	0.250292	0.2493	0.00455	0.000034	-9.627115
0.512	0.144		0.015616	0.00379	0	0.74615	0	
0.322	-0.150	-0.469	0.849707	0.583362	0.5797	0.00005	0.000001	-6.559478
0.775	0.357		0.039513	0.002012	0.00195	0.4183	0	
0.538	-0.014	-0.067	0.8482	0.253876	0.2422	0.0113	0.00001	-6.562127
0.540	0.124		0.019232	0.054216	0.0415	0.705	0	
0.372	-0.472	0.148	0.96838	0.889024	0.76525	0.1206	0.000069	-31.229412
1.048	0.094		0.301108	0.134006	0.00915	0.105	0	
0.177	-0.472	-0.461	0.98793	0.996215	0.73765	0.25855	0.000052	-82.195549
0.526	0.035		1.672039	0.263289	0.0028	0.001	0	
0.979	-0.038	-0.956	0.984642	0.412223	0.1802	0.0001	0.000559	-62.92566
0.745	0.088		2.004275	0.29575	0.0045	0.8152	0	
0.394	-0.701	0.418	0.925512	0.961157	0.96095	0	0.000015	-12.864097
0.392	0.480		0.090085	0	0	0.03905	0	
0.378	-0.649	-0.206	0.805579	0.955522	0.806	0.14995	0.000056	-4.900063
0.576	0.119		0.395147	0.161944	0.0126	0.03145	0	
0.459	-0.340	0.803	0.931337	0.718619	0.7227	0	0.000006	-14.490208
0.512	0.310		0.01619	0	0	0.2773	0	
0.209	-0.065	-0.909	0.999918	0.487582	0	0	0.120222	-10115.29498
0.405	0.206		2.297857	0	0	1	0	
0.425	-0.551	-0.928	0.846884	0.895718	0.9002	0.0004	0.000186	-5.938984
0.570	0.267		0.463193	0.023564	0.01415	0.08525	0	
0.187	-0.476	-0.900	0.958729	0.994485	0.9943	0	0.000006	-19.978996
0.121	0.109		2.043649	0	0	0.0057	0	
0.098	-0.602	0.385	0.977339	1	1	0	0.000015	-38.884895
0.898	0.803		0.172552	0	0	0	0	
0.190	-0.592	-0.352	0.970621	0.999053	0.79915	0.2001	0.000013	-33.088211
0.365	0.075		2.733268	0.194821	0.0005	0.00025	0	
0.352	-0.112	-0.760	0.956202	0.499804	0.00495	0	0.00109	-17.260636
0.698	0.483		1.259836	0.004152	0.00015	0.9949	0	
0.626	-0.460	0.366	0.941089	0.715838	0.61685	0.10105	0.000004	-16.958135
0.192	0.005		0.046438	0.109271	0.012	0.2701	0	
0.387	-0.473	-0.782	0.915861	0.878681	0.88185	0	0.00002	-11.738809
0.598	0.190		0.068834	0.000419	0.0002	0.11795	0	
0.153	-0.287	0.478	0.997543	0.969231	0.8029	0.1647	0.000002	-406.510721
0.259	0.013		1.486356	0.163763	0.0004	0.032	0	
0.364	-0.037	-0.442	0.951406	0.251422	0.24095	0.00015	0.000054	-20.480518
0.374	0.057		0.071044	0.002689	0.0025	0.7564	0	
0.121	-0.336	-0.011	0.996788	0.997204	0.99675	0	0.000004	-302.501664
0.160	0.042		0.953497	0	0	0.00325	0	
0.102	-0.121	-0.053	0.951675	0.868896	0.78835	0.077	0.000049	-20.250323
0.576	0.089		0.504098	0.087224	0.0147	0.11995	0	

Table A.3: (continued)

Simulated Parameters					Estimated values			
σ_1	μ_1	ρ	β	d_1	p_1	p_{12}	V_{D^*}	K
σ_2	μ_2		Γ	d_2	p_2	q	p_{D^*}	
0.296	-0.976	0.832	0.926368	0.999512	0.9992	0	0.000013	-12.407685
0.205	0.196		0.487608	0	0	0.0008	0	
0.410	-0.481	-0.190	0.987634	0.865676	0.85905	0.0005	0.000006	-80.698622
0.440	0.030		0.105716	0.000461	0.0001	0.14035	0	
0.123	-0.554	-0.167	0.914966	0.999997	1	0	0.000032	-9.568823
0.250	0.495		0.427829	0	0	0	0	
0.440	-0.405	0.002	0.959651	0.790649	0.76705	0.01775	0.000007	-24.683321
0.470	0.045		0.098316	0.023626	0.00435	0.21085	0.00005	
0.272	-0.348	0.007	0.986242	0.890243	0.88885	0.0011	0.000002	-72.508835
0.272	0.020		0.187625	0.000677	0	0.11005	0	
0.261	-0.331	-0.220	0.919727	0.889354	0.88655	0	0.000006	-12.351503
0.264	0.130		0.066208	0	0	0.11345	0	
0.111	-0.796	0.637	0.956438	1	1	0	0.000021	-7.66877
0.292	0.457		3.611563	0.000015	0	0	0	
0.965	-0.006	0.016	0.963167	0.112257	0.1034	0.00395	0.000009	-27.141437
0.671	0.036		0.003397	0.035403	0.02985	0.8628	0.0001	
0.324	-0.308	-0.693	0.85767	0.805124	0.77855	0.02455	0.000003	-6.998438
1.023	0.182		0.018137	0.080137	0.05505	0.14185	0	
0.150	-0.343	0.071	0.949181	0.988981	0.9871	0	0.000012	-17.486036
0.192	0.287		0.422932	0	0	0.0129	0	
0.104	-0.192	-0.933	0.956281	0.966946	0.93815	0.03035	0.000003	-22.851592
0.278	0.014		0.04877	0.060263	0.02735	0.00415	0	
0.248	-0.689	0.843	0.937077	0.997263	0.91625	0.08115	0.000141	-15.398857
0.547	0.106		0.583373	0.083186	0	0.0026	0	
0.371	-0.413	0.233	0.834301	0.854799	0.63195	0.21895	0.000004	-5.815521
0.343	0.025		0.719305	0.239266	0.01685	0.13225	0	
0.335	-0.013	-0.081	0.953395	0.169045	0.1687	0	0.000009	-21.339524
0.667	0.203		0.025599	0	0	0.8313	0	
0.461	-0.053	-0.979	0.981756	0.323815	0	0	0.067062	-46.840294
0.788	0.311		0.741642	0.002092	0	1	0	
0.396	-0.624	0.165	0.844172	0.939713	0.76515	0.17115	0.000043	-6.272032
0.864	0.101		0.161533	0.174709	0.00665	0.05705	0	
0.386	-0.261	0.042	0.906023	0.690356	0.6497	0.0386	0.000012	-10.618591
0.305	0.041		0.038862	0.054846	0.0153	0.2964	0	
0.251	-0.278	0.899	0.95682	0.848623	0.84455	0.00075	0.000016	-23.066832
0.413	0.064		0.066838	0.000747	0	0.1547	0	
0.443	-0.437	-0.139	0.975908	0.813064	0.73875	0.0712	0.000004	-41.457896
0.452	0.001		0.074635	0.096233	0.02545	0.1646	0	
0.626	-0.056	-0.910	0.99143	0.365143	0.0091	0.00045	0.000472	-114.818499
1.099	0.179		1.267658	0.256815	0.1206	0.86985	0	
0.473	-0.073	-0.794	0.974552	0.352478	0.04025	0	0.000367	-37.551632
0.711	0.202		0.622831	0.015352	0.0007	0.95905	0	
0.625	-0.256	-0.668	0.886866	0.54567	0.5381	0.0107	0.000009	-8.799534
0.083	0.009		0.124629	0.10144	0.0878	0.3634	0	
0.361	-0.245	0.585	0.936232	0.687768	0.6847	0	0.000003	-15.635109
0.325	0.263		0.01122	0	0	0.3153	0	
0.046	-0.279	-0.027	0.995661	1	0.6629	0.3371	0.000004	-229.769504

Table A.3: (continued)

Simulated Parameters					Estimated values			
σ_1	μ_1	ρ	β	d_1	p_1	p_{12}	V_{D^*}	K
σ_2	μ_2		Γ	d_2	p_2	q	p_{D^*}	
0.680	0.014		1.820531	0.33174	0	0	0	
0.184	-0.298	0.597	0.968629	0.945247	0.6497	0.29265	0.000051	-31.207432
0.371	0.024		2.82349	0.294991	0.0011	0.05655	0	
0.410	-0.011	-0.612	0.920236	0.258301	0.0037	0	0.000045	-9.937506
1.155	0.689		0.532974	0.009139	0.00215	0.99415	0	
0.412	-0.077	0.010	0.931187	0.346851	0.34745	0	0.000007	-14.502576
0.408	0.151		0.012166	0	0	0.65255	0	
0.182	-0.415	-0.360	0.948981	0.988666	0.7557	0.2331	0.000037	-19.10033
0.767	0.056		0.786916	0.237115	0.00605	0.00515	0	
0.478	-0.918	0.412	0.97939	0.971979	0.97045	0	0.000002	-48.092394
0.582	0.132		0.073994	0	0	0.02955	0	

Bibliography

Aase, K. K. (2002): Equilibrium Pricing in the presence of Cumulative Dividends following a diffusion. *Mathematical Finance* 12(3), 173–198.

Abramowitz, M., and I. Stegun (1964): *Handbook of Mathematical Functions with formulas, graphs and mathematical tables.* Washington: National Bureau of Standards.

Alili, L., and A. E. Kyprianou (2004): Some remarks on first passage of Lévy Processes, the American Put and Pasting Principles. Preprintavailable at `http://www.math.uu.nl/people/kypriano/pubs.html`

Aquilina, J., and L. C. G. Rogers (2004): The squared Ornstein-Uhlenbeck market. *Mathematical Finance* 14(4), 487-513.

Arrow, K., and G. Debreu (1954): Existence of an Equilibrium for a Competitive Economy. *Econometrica* 22(3), 265-290.

Asmussen, S., Hojgaard, B., and M. Taksar (2000): Optimal risk control and dividend distribution policies. Example of excess-of-loss reinsurance for an insurance corporation. *Finance and Stochastics* 4, 299-324.

Balakrishnan, N., Johnson, N. L., and S. Kotz (2000): *Continuous multivariate distributions. Vol. 1, Models and applications.* Wiley series in probability and statistics, 2nd ed., Wiley: New York.

Bally, V., and G. Pagès (2003): A quantization algorithm for solving multi-dimensional discrete-time optimal stopping problems. *Bernoulli* 9(6), 1003-1049.

Basak, S. (1995): A General Equilibrium Model of Portfolio Insurance. *Review of Financial Studies* 8(4), 1059-1090.

Basak, S., and D. Cuoco (1998): An Equilibrium Model with Restricted Stock Market Participation. *Review of Financial Studies*, Summer 1998, 11(2), 509-541.

Bermudez, A., and M. R. Nogueiras (2003): Numerical Solution of two-factor models for valuation of financial derivatives. *Mathematical Models and Methods in Applied Sciences.* Forthcoming.

Bermudez, A., and N. Webber. (2003): An asset based model of defaultable convertible bonds with endogenized recovery. Working Paper, Cass Business School, City University, London, available at `http://www.cass.city.ac.uk/facfin/facultypages/nwebber`

Berridge, S. (2004): Irregular Grid Methods for Pricing High-Dimensional American Options. *CentER Dissertation Series No. 131*, Tilburg University, the Netherlands.

Bertoin, J. (1996): *Lévy Processes.* Cambridge: Cambridge University Press.

Bertzekas, D. P. (1976): *Dynamic Programming and Stochastic Control.* Vol. 125 in the Series "Mathematics in Science and Engineering". Academic Press, New York.

Black, F., and J. C. Cox (1976): Valuing corporate securities: some effects of bond indenture provisions. *Journal of Finance* 31(2), 351-367.

Breeden, D. T. (1979): An Intertemporal Asset Pricing Model with Stochastic Consumption and Investment Opportunities. *Journal of Financial Economics* 7, 265-296.

Burda, M., and C. Wyplosz (1997): *Macroeconomics. A European Text.* Oxford: Oxford University Press, 1997.

Constantinides, G. M. (1982): Intertemporal Asset Pricing with Heterogeneous Consumers and Without Demand Aggregation. *Journal of Business* 55(2), 253-267.

Cottle, R.W., Pang, J.S., and R.E. Stone (1992): *The linear Complementarity Problem.* Academic Press, San Diego.

Cox, J. C., J. E. Ingersoll, and S. A. Ross (1985a). An Intertemporal General Equilibrium Model of Asset Prices. *Econometrica* 53(2), 363–384.

Cox, J. C., J. E. Ingersoll, and S. A. Ross (1985b): A Theory of the Term Structure of Interest Rates. *Econometrica* 53(2), 385–408.

Dempster, M. A. H., and J. P. Hutton (1999): Pricing American stock options by linear programming. *Mathematical Finance* 9(3), 229-254.

Detemple, J., and A. Serrat (2003): Dynamic Equilibrium with Liquidity Constraints. *Review of Financial Studies* 16(2), 597-629.

Duffie, D. (1986): Stochastic Equilibria: Existence, Spanning number, and the 'No Expected Financial Gains from Trade' Hypothesis. *Econometrica* 54(5), 1161–1184.

Duffie, D., and C. Huang (1985): Implementing Arrow-Debreu equilibria by continuous trading of a few long-lived securities. *Econometrica* 53(6) 1337–1356.

Duffie, D., and D. Lando (2001): Term structures of credit spreads with incomplete accounting information. *Econometrica* 69(3) 633-664.

Duffie, D., and K. J. Singleton (1999): Modeling Term Structures of Defaultable Bonds. *Review of Financial Studies* 12(4), 687-720.

Duffie, D., and W. Zame (1989): The Consumption-based Capital Asset Pricing Model. *Econometrica* 57(6), 1279–1297.

Giesecke, K. (2002): Correlated Default with incomplete information. Working Paper, Cornell University. To appear in *Journal of Banking and Finance.*

Giesecke, K. (2003): Successive correlated defaults: pricing trends and simulation. *Working Paper, Cornell University.*

Hara, C., and C. Kuzmics (2002): Representative Consumer's Risk Aversion and Efficient Risk-Sharing Rules. Working Paper, Faculty of Economics and Politics, University of Cambridge.

Hilberink, B., and L. C. G. Rogers (2002): Optimal capital structure and endogenous default. *Finance and Stochastics* 6, 237-263.

Hobson, D. G. and L. C. G. Rogers (1998): Complete Models of Stochastic Volatility. *Mathematical Finance* 8, 27–48.

Hojgaard, B., and M. Taksar (1999): Controlling risk exposure and dividend payout schemes: insurance company example. *Mathematical Finance* 9(2), 153-182.

Huang, C. (1987): An Intertemporal General Equilibrium Pricing Model: The case of Diffusion Information. *Econometrica* 55(1), 117-142.

Jarrow, R. A., Lando, D., and S. M. Turnbull (1997): A Markov Model for the Term Structure of Credit Risk Spreads. *Review of Financial Studies* 10(2), 481-523.

Jarrow, R. A., and S. M. Turnbull (1995): Pricing Derivatives on Financial Securities Subject to Credit Risk. *Journal of Finance* 50(1), 53-85.

Jarrow, R. A., and F. Yu (2001): Counterparty Risk and the Pricing of Defaultable Securities. *Journal of Finance* 56(5), 1765-1799.

Kadam, A., P. Lakner, and A. Srinivasan (2004): Perpetual Call Options with Non-Tradability. To appear in *Optimal Control Applications and Methods.*

Karatzas, I., J. P. Lehoczky, S. P. Sethi and S. E. Shreve (1986): Explicit Solution of a general Consumption / Investment problem. *Mathematics of Operations Research* 11(2), 261-294.

Karatzas, I., J. P. Lehoczky, and S. E. Shreve (1990): Existence and Uniqueness of Multi-Agent Equilibrium in a Stochastic, Dynamic Consumption/Investment Model. *Mathematics of Operations Research* 15(1), 80–128.

Karatzas, I., P. Lakner, J. P. Lehoczky, and S. E. Shreve (1991): Equilibrium in a Simplified Dynamic, Stochastic, Economy with Hetergoeneous Agents; in *Stochastic Analysis*, E. Mayer-Wolf, E. Merzbach, A. Shwartz, eds. Boston: Academic Press, 245-272.

Khadem, V., and W. Perraudin (2001): Default hazards and the term structure of credit spreads in a duopoly. Working Paper, Birkbeck College London, available at `http://wp.econ.bbk.ac.uk/respap`.

Kushner, H. J. and P. Dupuis (2001): *Numerical Methods for Stochastic Control Problems in Continuous Time.* Springer-Verlag: New York, 2nd edition.

Kwok, Y. K. (1998): *Mathematical Models of Financial Derivatives.* Springer: Singapore.

Lando, D. (1998): On Cox Processes and Credit Risky Securities. *Review of Derivatives Research* 2, 99-120.

Leland, H. E. (1994): Corporate Debt Value, Bond Covenants, and Optimal Capital Structure. *Journal of Finance* 49(4), 1213-1252.

Leland, H. E. and K. B. Toft (1996): Optimal Capital Structure, Endogenous Bankruptcy, and the Term Structure of Credit Spreads. *Journal of Finance* 51(3), 987-1019.

Li, D. X. (2000): On default correlation: A Copula function approach. *Journal of Fixed Income* 9(4), 43-54, March.

Lucas, R. E., Jr. (1978): Asset Prices in an Exchange Economy. *Econometrica* 46(6), 1429–1445.

Merton, R. C. (1969): Lifetime Protfolio Selection under Uncertainty: The Continuous-Time Case. *Review of Economics and Statistics* 51(3), 247-257.

Merton, R. C. (1974): On the Pricing of Corporate Debt: The Risk Structure of Interest Rates. *Journal of Finance* 29(2), 449-470.

Miao, J., and N. Wang (2004): Investment, Hedging and Consumption Smoothing. Working Paper, Boston University, July. Available at `http://people.bu.edu/miaoj/research.htm`.

Muller, K. E. (2001): Computing the confluent hypergeometric function, $M(a, b, x)$. *Numerische Matematik* 90, 176–196, May.

Nelson, D., and K. Ramaswamy (1990): Simple binomial processes as diffusion approximations in binomial models. *Review of Financial Studies* 3(3), 393-430.

Oberman, A. (2003): Convergent difference schemes for degenerate elliptic and parabolic equations: Hamilton-Jacobi equations and Free Boundary problems. Submitted to *SINUM*, available at `http://www.ma.utexas.edu/~oberman/`.

Oberman, A. and T. Zariphopoulou (2003): Pricing early exercise contracts in incomplete markets. *Computational Management Science* 1(1), 75-107.

Revuz, D., and M. Yor (2001): *Continuous Martingales and Brownian Motion.* Springer: Berlin.

Rogers, L. C. G. (1989): A guided tour through excursions. *Bull. London Math. Soc.* 21, 305-341.

Rogers, L. C. G. (2002): Monte Carlo valuation of American Options, *Mathematical Finance* 12(3), 271–286.

Rogers, L. C. G., and D. Williams (2000): *Diffusions, Markov Processes and Martingales, Vols. I and II.* Cambridge: Cambridge University Press.

Sato, K. (1999): *Lévy Processes and Infinitely Divisible Distributions.* Cambridge: Cambridge University Press.

Schönbucher, P. J., and D. Schubert (2001): Copula-dependent default risk in intensity models. Bonn University, Working paper, available at `http://www.schonbucher.de`.

Schroder, M. (1989): Computing the Constant Elasticity of Variance Option Pricing Formula. *Journal of Finance* 44(1), 211-219.

Sundaresan, M. (1984): Consumption and Equilibrium Interest Rates in Stochastic Production Economies. *Journal of Finance* 39(1) 77–92.

Wang, J. (1995): The Term Structure of Interest Rates in a Pure Exchange Economy with hetergoeneous investors. Working Paper No. W5172, MIT and NBER, Cambridge, Massachusetts.

Zhou, C. (2001): An Analysis of Default Correlations and Multiple Defaults *Review of Financial Studies* 14(2), 555-576.